On the Use of Sharpe's Index in Evolutionary Portfolio Optimization Under Markowitz's Model

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Abstract. Portfolio optimization is a problem that lends itself naturally to multiobjective approaches, e.g., aimed to maximize the return of the investment, simultaneously minimizing the risk. The selection of an actual portfolio requires exercising a decision-making process on the set of efficient solutions thus obtained. In this work we consider the case in which knowledge of this selection criterion is available, and used within the optimizer. We use Sharpe's index, a measure of excess return per unit of risk, for this purpose. It is shown that a multi-start single-objective evolutionary algorithm based on this index can provide a better coverage of the relevant regions of the Pareto front than state-of-the-art multiobjective evolutionary algorithms. An extensive experimental analysis is conducted using real data from a Latin American stock exchange.

1 Introduction

Portfolio optimization is a conspicuous problem in the area of financial management. Broadly speaking, it amounts to determining an adequate distribution of investments, such that an acceptable economic return is obtained, and good risk diversification is achieved. Obviously, the extent to which a particular return is considered acceptable or a certain risk diversification is good depends on the profile of the investor. There are a number of theoretical studies regarding the risk/performance relation. Among these, Markowitz's model [11] has become an essential theoretical reference for portfolio selection.

Markowitz's model is based on the rational behavior of the investor, who tries to maximize her profit and rejects the risk. The collection of portfolios offering a combination of risk/profitability such that no higher profit can be obtained without increasing the risk as well is termed the *efficient frontier*, and once known the investor can select her optimal portfolio according to her preferences. Of course, the determination of this efficient frontier (or Pareto front) is by no means an easy task in general. Fortunately, powerful optimization techniques can be used for this purpose, such as for example multiobjective evolutionary algorithms (MOEAs) [5, 3].

MOEAs have been deployed on portfolio optimization problems in numerous occasions, e.g., see [17, 15, 7] among other works. However, in this work we are specifically concerned not just about the calculation of a quasi-optimal Pareto front, but also on the subsequent decision-making process. In a recent work [4] we have analyzed the performance of several MOEAs in light of a very precise selection criterion, namely Sharpe's index. This index measures how much excess profit per risk unit delivers a certain portfolio, and is parameterized by a value that indicates the observed (or desired) return of a risk-free portfolio. An important consequence of the use of this selection criterion is the fact that specific regions of the Pareto front turn out to be more relevant, and hence algorithmic comparisons based on absolute multiobjective performance (as measured by standard quality indicators) do not necessarily coincide with the relative performance of selected solutions. This fact lead us to consider the inclusion of this selection criterion within the optimization process. This is specifically interesting when the parameter determining the risk-free return is not known or given in advance. We will assess the performance of a multistart EA based on this selection index, and analyze the quality of the results obtained in a variety of scenarios, with special emphasis in the comparison to state-of-the-art multiobjective EAs.

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2 Background

As state before, we consider an investment scenario in which an investor wishes to maximize profitability and minimize risk. This will be formalized within Markowitz's model in next subsection.

2.1 Markowitz's Model

Markowitz's model [11] assumes that the future performance that a specific investment can offer can be determined from both experience and investigation. Two main components have to be taken into account: profitability and risk to be assumed by the investor. The overall risk of the portfolio is defined as a weighted quadratic combination of the covariances of the assets included in it, i.e.,

$$\sigma^2(\vec{R}|\vec{W}) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}(\vec{R}) \tag{1}$$

where $\vec{W} = \{w_i\}$, $1 \leqslant i \leqslant n$, is a vector comprising the fraction of the budget allocated to each asset ($w_i \geqslant 0$), and $\sigma_{ij}(\vec{R})$ is the covariance of the performance of the i-th asset and the j-th asset, defined as:

$$\sigma_{ij}(\vec{R}) = \sum_{t=1}^{T} \frac{[R_{it} - E(R_i)]\,[R_{jt} - E(R_j)]}{T} \tag{2}$$

where $\vec{R} = \{R_{it}\}$, $1 \leqslant i \leqslant n$, $1 \leqslant t \leqslant T$, is a matrix containing the profitability of each asset at each time interval t, $E(R_i)$ is the mean profitability of the i-th asset, and T is the number of intervals in the time horizon.

Similarly to the risk, the profitability $E(\vec{R}|\vec{W})$ of a portfolio is defined as the weighted average of the assets involved, i.e.,

$$E(\vec{R}|\vec{W}) = \sum_{i=1}^{n} w_i E(R_i) \tag{3}$$

Once the profitability and risk of a portfolio is defined, there remains the issue of determining which portfolio, among all available possibilities, should be preferred. This is tackled in next subsection.

2.2 Sharpe's Index

Generally speaking, the investor looks for the curve of utility with $E(\vec{R}|\vec{W}) = \infty$ and $\sigma^2(\vec{R}|\vec{W}) = 0$, but this not a realistic option as this curve is limited by the existing assets that never have this nature. We note that for the assets without risk (i.e., those with null profit-variance), the utility is equal to the expected profitability because there is no penalization due to the risk.

To evaluate the quality of a portfolio we have to define a measure that accounts for both the profitability and the risk of the assets involved. Such a measure can also allow the comparison between different portfolios. To this end, we have considered Sharpe's index [13], that determines performance according to the ratio between excess profitability and risk. More precisely,

$$S(\vec{R}|\vec{W}) = \frac{E(\vec{R}|\vec{W}) - R_0}{\sigma(\vec{R}|\vec{W})} \tag{4}$$

where R_0 is the performance of a portfolio without risk. $E(\vec{R}|\vec{W}) - R_0$ is therefore the excess performance (that is, the extra profit obtained by taking some risks), which is divided by the risk of the portfolio (measured as the standard deviation of returns). Basically, the index indicates how much extra performance is expected with respect to the risk. The higher the value returned is, the higher the success of the fund management is.

3 Evolutionary Portfolio Selection and Sharpe's Index

The portfolio selection problem posed in previous section will be tackled with evolutionary algorithms (EAs). This can be done from a multiobjective perspective, that is, finding a quasi-optimal set of efficient solutions and selecting one of them using a specific decision-making procedure (maximizing Sharpe's index in our case). Alternatively, this selection criterion can be directly embedded within the fitness function. Both possibilities are described next.

3.1 Optimization Setting

As stated before, Markowitz's model is based on the assumption the investor abhors risk, which can be represented as the variability of returns for a certain investment. At the same time, she wants to maximize her profits. Hence, we can define a portfolio as efficient if it achieves the profit sought by the investor at the minimum risk. This consideration leads naturally to a multiobjective scenario in which Pareto-optimal portfolios are sought, i.e., portfolios whose profitability cannot be increased without increasing the risk as well (and vice versa, the risk cannot be reduced without decreasing the expected return too). This set of efficient portfolios can be calculated by solving the bi-objective problem $\left\{\min \sigma^2(\vec{R}|\vec{W}), \max E(\vec{R}|\vec{W})\right\}$ subject to $\sum_{i=1}^{n} w_i = 1$.

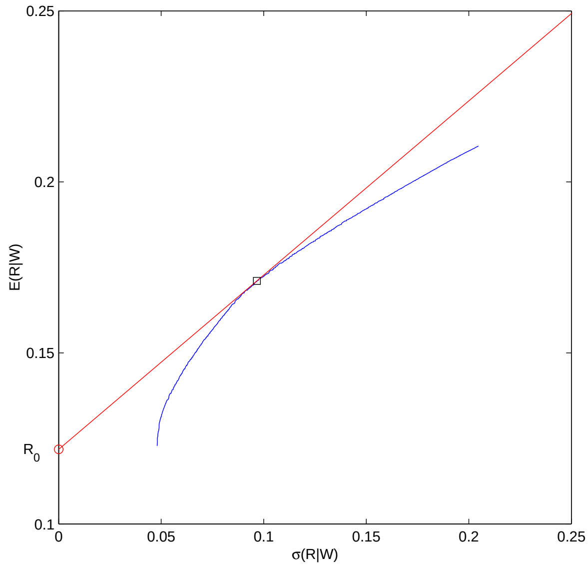

Figure 1. Selection of a solution from the Pareto front using Sharpe's index. The reference risk-free return R_0 is marked with an open circle on the Y axis, whereas the selected solution from the Pareto front (shown in blue) is marked with a square. In this example, both the Pareto front and the reference point R_0 correspond to real data used in the experimentation (mixed funds, see Sect. 4).

Subsequently, once the optimal Pareto set (or a good approximation to this set) has been calculated, the decision-making criterion can be exercised to extract a single preferred solution from this set. In our case, we consider the use of Sharpe's index for this purpose, as stated before.

It is interesting to note the geometrical interpretation of this decision-making procedure based on Sharpe's Index. Since this index is the ratio between excess profit and risk, all profit/risk pairs located along a straight line running through $(0, R_0)$ correspond to hypothetical portfolios with the same value for Sharpe's index (which is the slope of this line, actually). Furthermore, only points below the optimal Pareto front may represent attainable portfolios. Therefore, the maximum value of Sharpe's index corresponds to the line of highest slope that runs through both the point $(0, R_0)$ and an actual portfolio. This corresponds to a tangent to the Pareto front as illustrated in Fig. 1.

This geometrical interpretation leads to an additional observation: given two portfolios $\vec{W}_1$ and $\vec{W}_2$, if the former has a higher value of Sharpe's index, then it is not dominated (in the Pareto sense) by the latter. This can be easily seen by noting that in order for $\vec{W}_1$ to be dominated by $\vec{W}_2$, it must be that $E(\vec{R}|\vec{W}_1) \leqslant E(\vec{R}|\vec{W}_2)$ and $\sigma^2(\vec{R}|\vec{W}_1) \geqslant \sigma^2(\vec{R}|\vec{W}_2)$, with at least one of the inequalities being strict. However, if this is the case, then $\vec{W}_2$ has a higher value of Sharpe's index, since it has a larger numerator and a smaller denominator in Eq. (4). This fact paves the way to defining a single-objective EA that directly tries to maximize Sharpe's index: by progressing toward better values of the index, the EA will also advance toward the Pareto front, ideally converging to the optimal tangent point. Of course, this ideal behavior does not have to happen necessarily, since there may be local optima in the search landscape along the way (as dictated by the risk/profit profile of attainable portfolios). We will return to this point in Sect. 4, when analyzing the experimental results.

3.2 Algorithmic Approaches

The multiobjective portfolio optimization problem has been tackled with three state-of-the-art MOEAs, namely NSGA-II (Non-dominated Sorting Genetic Algorithm II) [6], SPEA2 (Strength Pareto Evolutionary Algorithm 2) [22] and IBEA (Indicator-Based Evolutionary Algorithm) [21]. The first two ones are the second-generation version of two previous algorithms (NSGA [14], and SPEA [23] respectively). As such, they rely on the use of elitism (an external archive of non-dominated solutions in the case of SPEA2, and a plus-replacement strategy –keeping the best solutions from the union of parents and offspring– in the case of NSGA-II). More precisely, the central theme in these algorithms is assigning fitness to individuals according to some kind of non-dominated sorting, and trying to preserve diversity among solutions in the non-dominated front. The third algorithm considered is IBEA, which is aimed to maximize some multiobjective performance indicator, and uses a replacement strategy that tries (in a greedy way) to optimize the value of this indicator for the current population. In this work, we have considered an IBEA based on the ε-indicator [19].

In addition to the multiobjective EAs mentioned before, we have also considered a single-objective EA aimed to optimizing Sharpe's index. This EA is termed SEA –after Sharpe's index-based EA– and has the advantage of being a simpler approach, since no archiving of solutions nor Pareto-based selection/replacement is necessary. More precisely, we have considered the use of binary tournament selection, and a plus replacement strategy.

In all algorithms considered, solutions –that is, a vector of rational values in the $[0, 1]$ range, indicating the fraction of the portfolio devoted to each fund– are represented as binary strings. Each fund is assigned 10 bits, yielding a raw weight $\bar{w}_i$. These weights are subsequently normalized as $w_i = \bar{w}_i / \sum_j \bar{w}_j$ to obtain the actual composition of the portfolio. Evaluation is done by computing the risk and return of the portfolio using the formulation depicted before. As to reproduction, we consider standard operators such as two-point crossover and bit-flip mutation.

4 Results

The data used in the experiments is taken from the Caracas Stock Exchange (*Bolsa de Valores de Caracas* - BVC), the only securities exchange operating in Venezuela. More precisely, we have considered data spanning from five up to eight years of stock trading. This time interval is large enough to be representative of the evolution of shares, and not too large to include irrelevant –for prediction purposes- data (the status of funds can fluctuate in the long term, commonly making old data useless for forecasting the future evolution of shares). According to this, our sample – $\sim 35,000$ daily prices of different mutual funds: fixed, variable, and mixed– comprises funds operating for at least five years and still available in the BVC [1]. To be precise, we have used weekly market data from year 1994 to year 2002, corresponding to 26 Venezuelan mutual funds: 12 fixed funds, 7 variable funds, and 7 mixed funds. Data up to year 2001 is used for training purposes, whereas data corresponding to the year 2002 will be used for testing the obtained portfolios with respect to an investment portfolio indexed in the BVC. The relative ratio of share values in successive weeks is calculated to compute the profitability of each fund. This is done for each week in the year, and subsequently averaged to yield the annual weekly mean and thus obtain the annual profit percentage. The covariance matrix of these profitability values is also computed, as a part of Markowitz's model.

Experiments have been done with the four algorithms described before, namely NSGA-II, SPEA2, IBEA, and SEA. For the first three techniques –the multiobjective EAs– we have utilized the PISA library (A Platform and Programming Language Independent Interface for Search Algorithms) [2]. In all cases, the crossover rate is $P_X = 0.8$, the mutation rate is $P_M = 1/\ell$, and the population size is 2ℓ, where ℓ is the total number of bits in a solution. The algorithms have been run for a maximum number of 100 generations. The number of runs per data set is 30.

The first part of the experimentation considers the use of the value $R_0 = 0.1218$ observed during the time window considered. Using this value, a single solution can be selected from the Pareto front obtained by each multiobjective EA. Likewise, the SEA can use this value to evolve portfolios with high values of Sharpe's index. The results are shown in Fig. 2. As it can be seen, while there is a certain amount of variability in the results provided by the MOEAs in each run, SEA does consistently provide a focused result, notably better (with statistical significance at the standard 0.05 level, using a Wilcoxon ranksum test [10]) than the remaining algorithms.

A natural question arises from these previous results, namely the extent to which SEA would be capable of beating the remaining multiobjective approaches should the value of R_0 be different. Notice that by varying this value, the tangent point to the Pareto front varies, and the convergence properties of SEA may be different. The MOEAs are not affected by this change though, since R_0 is only used in the decision-making process, after the algorithm has been run.

To investigate this issue, we have considered values of R_0 ranging from 0 to $R_0^{\max}$ (0.39, 0.20 and 0.40 for fixed, mixed and variable funds respectively; these values have been chosen by considering the highest profit attained in the grand overall Pareto front). To allow a fair comparison between SEA and the MOEAs, 30 equally spaced values from this range have been selected, and a single run of SEA has been done on each of them. The underlying question we want to address is whether it is better to devote the allotted computational effort (in this case, 30 runs of 10,000 evaluations each) to 30 multi-objective runs, or to 30 different single-objective runs.

The first aspect of the results we have analyzed is the structure of the Pareto front attained in each case. Notice that by archiving the solutions obtained by SEA in each run, we can actually build a non-dominated front, even when each of the independent runs was mono-objective. Of course, it cannot be expected in principle that the front provided this way by SEA is competitive in a general sense with that of the MOEAs, but an analysis of this front can anyway provide interesting information on the behavior of SEA. This analysis has been conducted using two well-known quality indicators: the hypervolume indicator [18] and the R_2 indicator [8] – see also [20]. The first one provides an indication of the region in fitness space that is dominated by the front (and hence the larger, the better), and the second indicator estimates the extent to which a certain front approximates another one (the true Pareto-optimal front if known, or a reference front otherwise). To be precise, we have considered the unary version of this indicator, taking the combined NSGA-II/SPEA2/IBEA/SEA Pareto front as reference set. Being a measure of distance to the reference set, the lower a R_2 value, the better. The lower right corner of the minimum box comprising this combined front has been used as reference point for hypervolume calculation.

Figs. 3 and 4 show the results for the experiments realized. In each case, the upper boxplot indicates the distribution of indicator values for the fronts obtained in each execution of the MOEAs (it does not make sense in the case of SEA, since each execution is going to be focused on a very specific region of the front, and hence the indi-

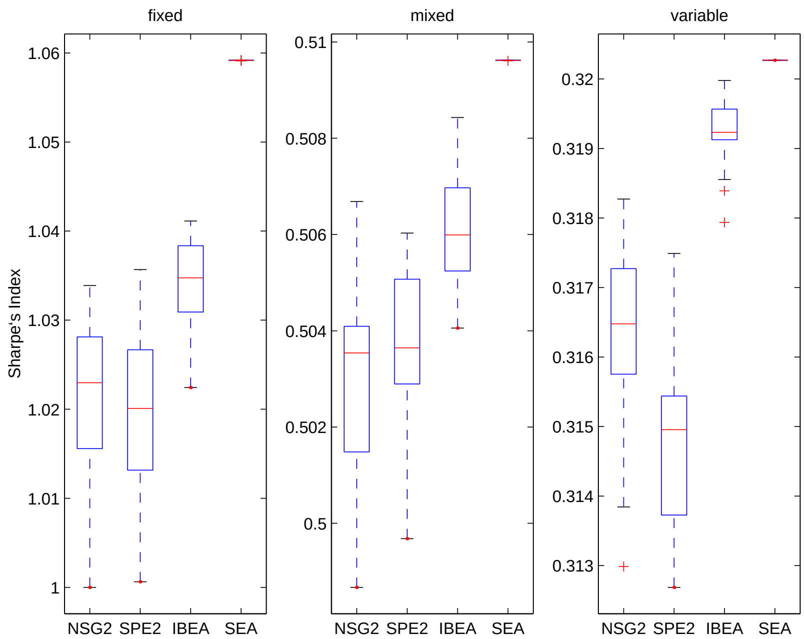

Figure 2. Boxplot of Sharpe's index values attained by NSGA-II, SPEA2, IBEA and SEA on fixed funds (right), mixed funds (middle) and variable funds (left). As usual, the boxes comprise the two middle quartiles of the distribution, the central line indicates the median of the distribution, the whiskers span 1.5 times the interquartile range, and outliers are indicated by a + symbol.

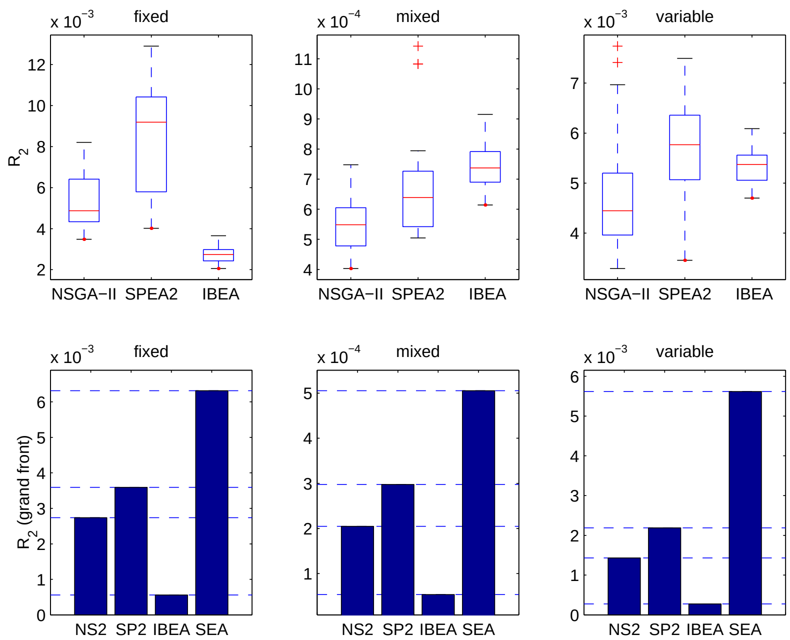

Figure 3. (Top) Boxplot of the R_2 indicator for NSGA-II, SPEA2 and IBEA. (Bottom) Indicator values for the grand fronts.

cator values are going to be poor and uninformative), and the lower bar graph indicates the indicator value for the grand front obtained aggregating the 30 runs of each algorithm.

Let us consider the R_2 indicator in first place (Fig. 3). As it can be seen, the grand front generated by IBEA is notably better than the remaining fronts; the grand front provided by SEA is on the contrary much worse than that of the other algorithms. This can be explained by the fact that SEA only focuses on the regions of the front that are tangent to a straight line originating at $(0, R_0)$ for the values of R_0 considered. The coverage of intermediate zones between these regions is weaker, and hence the grand SEA front exhibits a larger distance to the grand overall front used as reference. Notice that IBEA also exhibits an uneven behavior in individual runs, but manages to provide a good coverage of the reference front when its

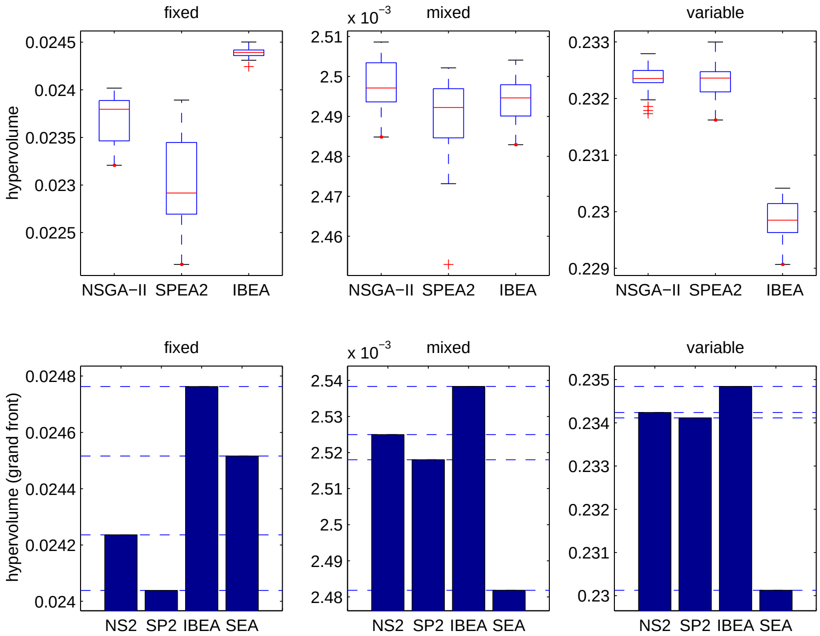

Figure 4. (Top) Boxplot of the hypervolume indicator for NSGA-II, SPEA2 and IBEA. (Bottom) Indicator values for the grand fronts (notice the range of values in the vertical axis).

30 runs are aggregated. As to the hypervolume indicator, it indicates that IBEA provides the best front too, but this time SEA does not appear to be much worse (actually, the difference is very small, as it can be seen by inspecting the range of values in the Y axis). This has an explanation: although SEA only achieves a good coverage of specific parts of the front, it is capable of advancing much deeper there, thus increasing the dominated hypervolume.

To see how the qualitatively different behavior of SEA with respect to the MOEAs affects its performance on different investment scenarios, the next step of the analysis focuses on the relative performance of selected solutions under different values of R_0. To be precise, for each algorithm we have considered its grand front, and selected the best solution (according to Sharpe's index) for 100 different values of R_0 (between 0 and the corresponding value of $R_0^{\max}$). Notice that this sample of values of R_0 is larger than that used in SEA runs, and includes many intermediate values not considered in the execution of this latter algorithm. Fig. 5 shows the results. Values of Sharpe's index have been normalized, dividing by the corresponding values returned by SEA. Therefore, values above (resp. below) 1.0 indicate better (resp. worse) values of Sharpe's index that those provided by SEA.

In general, there is an intermediate range of R_0 values for which all algorithms perform similarly. This range is larger in the case of mixed funds than for fixed or variable funds. In any case, NSGA-II and SPEA2 perform below SEA, and start to diverge quickly for large values of R_0. IBEA is capable of performing similarly to SEA on a broader range of R_0 values, but again falls below for larger values. In all cases this indicates that SEA has a better coverage of the upper-right region of the front (high profit, high risk), containing the solutions selected for increasing values of R_0. It is also interesting to note the behavior of SEA for low values of R_0 on fixed funds. There is a small interval of R_0 values where the MOEAs perform better than SEA, which converges toward a suboptimal region of the front (cf. Sect. 3.1). Subsequent multiple runs of SEA on this particular interval of values indicate that this suboptimal region has a strong basin of attraction when Sharpe's index is used as fitness function. Only around 5% of SEA runs are capable of converging toward the best region identified by the MOEAs for this particular case. However, this behavior does not take place outside this small interval of R_0 values, nor in the remaining data sets.

5 Conclusions

Portfolio optimization is a natural arena for multiobjective optimizers, which are both powerful and flexible enough to deal with this kind of problems. This is specifically true if the optimization process is done in absence of knowledge on the particular decision-making process that will take place afterwards, in order to select a solution from the Pareto front. However, if this knowledge is available, the case for a full-fledged multiobjective approach is not so strong, at least to the extent that this multiobjective approach treat all points in the Pareto front on the same basis. In this sense, we have shown that a multi-start single-objective EA based on Sharpe's index can outperform state-of-the-art MOEAs on several portfolio optimization instances, and on a different range of investment scenarios, providing a selective coverage of interesting regions of the Pareto front.

This does not imply by any means that MOEAs cannot be useful for this optimization task. For example, we have obtained evidence that the SEA can in certain circumstances face difficulties to locate the optimal region (according to the decision-making procedure chosen) of the Pareto front for a given profit objective. Although multiple additional runs may solve this problem, further analysis is clearly required here. We plan to carry on a deeper study of the fitness landscape in this scenario to characterize this situation. Future work will be also directed to analyze other variants of the problem where additional constraints are introduced, e.g., cardinality constraints, lot sizes, etc. This line of research is underway. Another line of future research concerns the measure of risk. We have focused on variance here, but there are other options. We may for example consider value at risk, i.e., the maximum loss that can take place at a certain confi-

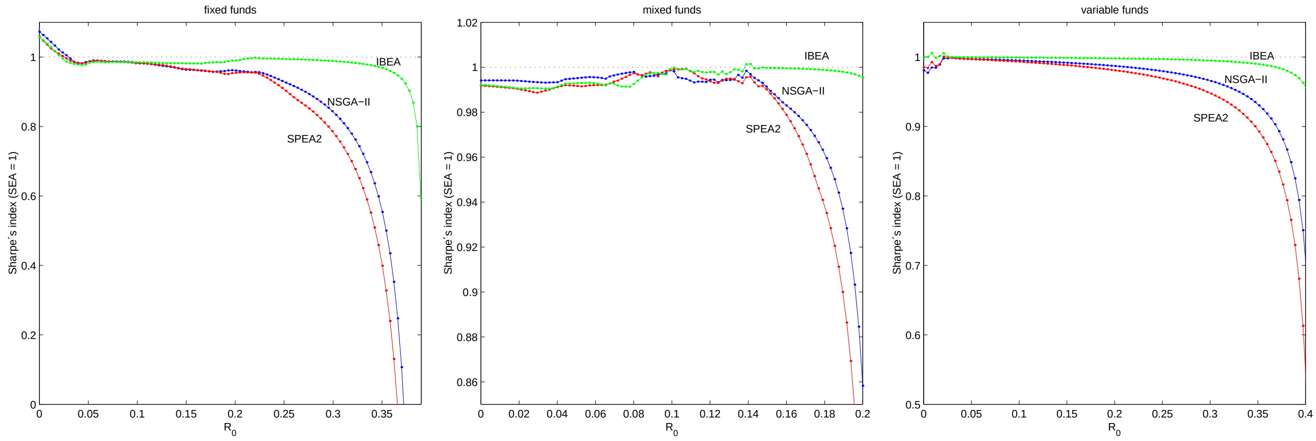

Figure 5. Normalized values of Sharpe's index provided by NSGA-II, SPEA2, and IBEA for different values of R_0.

dence level. A related measure is the conditional value at risk, namely the expected shortfall in the worst $q\%$ of cases, where q is a parameter. Other possible measures are Jensen index [9], Treynor index [16], or models emanating from capital asset pricing theory (CAPM) [12], among others.

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