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Elementary landscape decomposition of the frequency assignment problem

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ABSTRACT

The Frequency Assignment Problem (FAP) is an important problem that arises in the design of radio networks, when a channel has to be assigned to each transceiver of the network. This problem is a generalization of the graph coloring problem. In this paper we study a general version of the FAP that can include adjacent frequency constraints. Using concepts from landscapes' theory, we prove that this general FAP can be expressed as a sum of two elementary landscapes. Further analysis also shows that some subclasses of the problem correspond to a single elementary landscape. This allows us to compute the kind of neighborhood information that is normally associated with elementary landscapes. We also provide a closed form formula for computing the autocorrelation coefficient for the general FAP, which can be useful as an *a priori* indicator of the performance of a local search method.

1. Introduction

We define a *landscape* for a combinatorial optimization problem as a triple (X, N, f), where $f : X \to \mathbb{R}$ defines the objective function and the *neighborhood operator* function $N(x)$ generates the set of points reachable from $x \in X$ in a single application of the neighborhood operator. If $y \in N(x)$ then we say that y is a neighbor of x. The landscape that is so induced can be used as a search space for optimization using local search. Without loss of generality, we can define f so as either to be minimized or maximized over X. In this work we assume that f is minimized.

Elementary landscapes are a special form of landscape which have a number of particular properties. One of these properties is that they fulfill the so-called *Grover's wave equation*:

$$\operatorname*{avg}_{y\in N(x)}\{f(y)\} = f(x) + \frac{\lambda}{d}\left(\bar{f} - f(x)\right)$$

where $d = |N(x)|$ is the size of the neighborhood, which we assume the same for all the solutions in the search space (*regular neighborhood*), $\bar{f}$ is the average solution evaluation over the entire search space, and λ is a problem characteristic constant. The wave equation also makes it possible to compute the average value of the fitness function f evaluated over all of the neighbors of x with a single function evaluation; we denote this average using $\operatorname{avg}\{f(y)\}_{y\in N(x)}$:

$$\operatorname*{avg}_{y\in N(x)}\{f(y)\} = \frac{1}{d}\sum_{y\in N(x)} f(y).$$

Other properties also follow. Assuming $f(x) \neq \bar{f}$ and $0 < \lambda/d < 1$ then one can show by simple algebra,

$$f(x) < \operatorname*{avg}_{y\in N(x)}\{f(y)\} < \bar{f} \quad \text{or} \quad f(x) > \operatorname*{avg}_{y\in N(x)}\{f(y)\} > \bar{f}.$$

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This means that all maxima are greater than $\bar{f}$ and all minima are smaller than $\bar{f}$ [1].

Grover [2] first showed that certain problems (the Traveling Salesman Problem, Graph Coloring, Min-Cut Graph Partitioning, Weight Partition, as well as Not-all-equal-SAT) have common and natural local search neighborhoods that can be modeled using the wave equation. Stadler [3] named this class of problems "elementary landscapes" and has explored various properties of elementary landscapes.

Whitley and Sutton [4] used a component model to explain why certain classes of elementary landscapes obey the wave equation. This involves decomposing the objective function f into a linear combination of cost components. The cost components sometimes take the form of a cost matrix; for example, the distance matrix that is used to calculate the distance between cities in the Traveling Salesman Problem is such that each distance is one component of the cost function. For an n city problem, the evaluation of one tour is a linear combination of n distances. In Graph Coloring, the components can also be defined to be the weights of a lower triangular cost matrix M, where $m_{i,j} = 0$ when $i \leq j$. In Graph Coloring the cost is usually 1 for each conflicted edge; however, we show in this paper that the Graph Coloring problem is still elementary when an arbitrary weight matrix is used to assign costs to conflicted edges.

If the landscape is not elementary, the objective function can be written as a sum of elementary functions [5]. This decomposition into elementary functions is well-known for some problems. This is the case for all the pseudo-boolean functions and MAX-k-SAT in particular [6]. The reader interested on landscapes' theory can find in [7] a nice survey by Reidys and Stadler.

The three main contributions of this paper are the following ones. First, we prove that the cost function of a generalized form of the Frequency Assignment Problem (FAP) can be written as the sum of at most two different elementary landscapes. We analyze the theoretical implications of this fact. Second, we show that special cases of the generalized FAP can be expressed as a single elementary landscape. And third, we provide a closed form formula for computing the *autocorrelation coefficient* of any instance of the generalized FAP in polynomial time. This coefficient has interesting applications in practice since it is an *a priori* indicator of the performance of a local search method based on the underlying neighborhood. We use the elementary landscape decomposition to derive the formula. This work extends the results presented in [8], where the authors proved that some particular versions of the FAP used here can be decomposed as a sum of two elementary landscapes.

The organization of the paper is as follows. In the next section we present the required background on landscapes' theory. Section 3 describes the FAP in detail. In Section 4 we prove that the cost function of a FAP is the sum of two elementary landscapes in the general case, and Section 5 derives the conditions under which the cost function is an elementary landscape. Section 6 presents a closed formula for computing the autocorrelation coefficient for any instance of the problem in polynomial time. We also provide the least upper bound and the greatest lower bound of the autocorrelation coefficient. Finally, Section 7 concludes the paper and outlines future work.

2. Background on landscapes' theory

In this section we present some fundamental results on landscapes' theory. Most of the results presented here can be found in previous work [7]. However, we highlight some observations that can be easily derived from well-known facts but are not present in the previous literature as far as we know.

Let X be a finite set of solutions, $f : X \to \mathbb{R}$ be a real-valued function defined on X and $N : X \to \mathcal{P}(X)$ the neighborhood operator. We can represent the neighborhood operator by its adjacency matrix

$$A_{xy} = \begin{cases} 1 & \text{if } y \in N(x) \\ 0 & \text{otherwise} \end{cases}$$

The degree matrix D is defined as the diagonal matrix

$$D_{xy} = \begin{cases} |N(x)| & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Any discrete function over the set of candidate solutions, e.g. f, can be characterized as a vector in $\mathbb{R}^{|X|}$. Any $|X| \times |X|$ matrix can be interpreted as a linear map that acts on vectors in $\mathbb{R}^{|X|}$. The Laplacian matrix of a neighborhood operator is defined as

$$\Delta = A - D.$$

The Laplacian matrix acts on function f as follows

$$\Delta f = \begin{pmatrix} \sum_{y \in N(x_1)} (f(y) - f(x_1)) \\ \sum_{y \in N(x_2)} (f(y) - f(x_2)) \\ \vdots \\ \sum_{y \in N(x_{|X|})} \big(f(y) - f(x_{|X|})\big) \end{pmatrix}.$$

The component x of this matrix–vector product can thus be written as:

$$(\Delta f)(x) = \sum_{y \in N(x)} (f(y) - f(x)) . \tag{1}$$

In this paper, we will restrict our attention to regular neighborhoods, where $|N(x)| = d > 0$ for a constant d, for all $x \in X$. When a neighborhood is regular, $\Delta = A - dI$. Stadler defines the class of elementary landscapes where the function f is an eigenvector (or eigenfunction) of the Laplacian up to an additive constant [3]. Formally, we have the following

Definition 1 (*Elementary Function and Landscape*)**.** Let (X, N, f) be a landscape and Δ the Laplacian matrix of the neighborhood operator N. The function f is said to be elementary if there exists a constant b, which we call *offset*, and an eigenvalue λ of $-\Delta$ such that $(-\Delta)(f - b) = \lambda(f - b)$. The landscape itself is elementary if f is elementary.

In the following we use $-\Delta$ instead of Δ, as usual in the literature, to work with non-negative eigenvalues. According to the previous definition, every elementary function, f, can be written as the sum of an eigenfunction of $-\Delta$, g, and a constant b, i.e., $f = g + b$. Taking into account basic results of linear algebra, it is not difficult to prove that if f is elementary with eigenvalue λ, $af + b$ is also elementary with the same eigenvalue λ. The next properties are a consequence of the particular characteristics of $-\Delta$.

Proposition 1. *Given the function $f : X \to \mathbb{R}$ and the Laplacian Δ defined on the neighborhood operator N the following properties hold:*

1. *If f is a constant function, i.e., $f(x) = b\ \forall x \in X$ for a constant b, then $(-\Delta) f = 0$ and f is eigenfunction of $-\Delta$ with eigenvalue $\lambda = 0$.*
2. *If f is elementary for the neighborhood N with eigenvalue λ, then there exists a constant b such that*

$$\operatorname*{avg}_{y \in N(x)}\{f(y)\} = f(x) + \frac{\lambda}{d}(b - f(x)) \tag{2}$$

where d is the size of the neighborhood.

Proof. For the first property we can use Eq. (1) and write:

$$(-\Delta f)(x) = \sum_{y \in N(x)} (f(x) - f(y)) = \sum_{y \in N(x)} (b - b) = 0.$$

This happens for each $x \in X$, so $-\Delta f = 0$ and it is an eigenfunction of $-\Delta$ with eigenvalue 0.
For the second property we again use Eq. (1) to write:

$$(\Delta f)(x) = \sum_{y \in N(x)} (f(y) - f(x)) = \sum_{y \in N(x)} f(y) - d f(x).$$

Dividing the previous equation by d we get:

$$\frac{1}{d}(\Delta f)(x) = \operatorname*{avg}_{y \in N(x)}\{f(y)\} - f(x). \tag{3}$$

Since f is elementary with eigenvalue λ, there exists a constant b such that $-\Delta(f - b) = \lambda(f - b)$. Then, we can write with the help of (3):

$$\frac{1}{d}(\Delta(f - b))(x) = \frac{1}{d}(\Delta f)(x) = \operatorname*{avg}_{y \in N(x)}\{f(y)\} - f(x) = -\frac{\lambda}{d}(f(x) - b)$$

where we used the first property to remove b from the first member. We can rewrite the two last members as

$$\operatorname*{avg}_{y \in N(x)}\{f(y)\} = f(x) + \frac{\lambda}{d}(b - f(x)). \quad \square$$

Eq. (2) is quite similar to Grover's wave equation. The only difference is the constant, which in Grover's equation is $b = \bar{f}$, where $\bar{f}$ is the average of the function f over the entire solution set X, that is, $\bar{f} = \left(\sum_{x \in X} f(x)\right)/|X|$. In the following we will prove that Grover's equation is valid if the neighborhood is *symmetric*. We say that a neighborhood N is symmetric if for all $x, y \in X$ it holds that $y \in N(x)$ implies $x \in N(y)$, that is, if y is neighbor of x then x is neighbor of y. As far as we know, Eq. (2) has not previously been reported in the literature. Its relevance comes from the fact that it is valid in all the regular neighborhoods (not only in the symmetric ones). For the symmetric neighborhoods the following lemma holds.

Lemma 1. *Let N be a symmetric neighborhood over the solution set X and Δ its Laplacian matrix. If f is an eigenvector of $-\Delta$ with $\lambda \neq 0$ then $\bar{f} = 0$.*

Proof. Two eigenvectors of a symmetric matrix with different eigenvalues are orthogonal. In Proposition 1 we proved that any constant function is an eigenvector of $-\Delta$ with eigenvalue $\lambda = 0$. Thus, if f is an eigenvector of $-\Delta$ with eigenvalue $\lambda \neq 0$ then f is orthogonal to any constant function. In particular, it is orthogonal to the function $(1, 1, \ldots, 1)$ and we can write:

$$\bar{f} = \frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|X|} (1, 1, \ldots, 1) f = 0. \quad \square$$

In the previous lemma the reader should notice the requirement $\lambda \neq 0$. In Proposition 1 we proved that constant functions are eigenvectors of $-\Delta$ with $\lambda = 0$. Now we can ask the opposite: are all the eigenvectors of $-\Delta$ with $\lambda = 0$ constant functions? The general answer is no. However, there exists a kind of neighborhood in which the answer to the previous question is yes. We say that a neighborhood N is connected if for each pair of solutions $x, y \in X$ we can find a finite sequence of solutions $x = x_1, x_2, \ldots, x_q = y$ such that $x_{i+1} \in N(x_i)$ for $i = 1, 2, \ldots, q-1$. If the neighborhood N is *connected* then the multiplicity of the eigenvalue $\lambda = 0$ is one [5], and this means that only constant functions are eigenvectors of $-\Delta$. With all the previous results we are ready to enunciate the following

Theorem 1 (*Grover's Wave Equation*)**.** *Let (X, N, f) be a landscape where the neighborhood, N, is regular and symmetric. Then, f is elementary if and only if there exists a constant λ such that the following expression holds*

$$\operatorname*{avg}_{y \in N(x)}\{f(y)\} = f(x) + \frac{\lambda}{d}\left(\bar{f} - f(x)\right) \quad \forall x \in X \tag{4}$$

and λ is the eigenvalue of f.

Proof. First we consider that the neighborhood is regular and symmetric, and that the function f is elementary. As we previously proved, this implies that there exists a constant b such that Eq. (2) holds. We only need to prove that this constant b is exactly $\bar{f}$. In the proof of Proposition 1 we saw that b is a constant for which $(-\Delta)(f - b) = \lambda(f - b)$. This means that $g = f - b$ is an eigenvector of $-\Delta$ with eigenvalue λ. If $\lambda = 0$, Eq. (4) trivially holds. If $\lambda \neq 0$ we know by Lemma 1 that $\bar{g} = 0$. Then, we can write: $\bar{f} = \bar{g} + b = b$, and Eq. (4) holds.

Now, let us consider that Eq. (4) is true. Then, we can multiply (4) by d and write $\sum_{y \in N(x)} f(y) = d\,f(x) + \lambda\left(\bar{f} - f(x)\right)$, which we can write in vectorial form as:

$$-\Delta f = \lambda(f - \bar{f}).$$

Since $-\Delta \bar{f} = 0$ we can write:

$$-\Delta(f - \bar{f}) = \lambda(f - \bar{f})$$

so f is elementary with eigenvalue λ. $\square$

From Grover's wave equation we conclude that in an elementary landscape there exists a linear relationship between the average of the function in the neighborhood of a solution and the value of the function in that solution. We now ask if the linear relationship is something exclusive for elementary landscapes or not. The following proposition positively answers this question.

Proposition 2. *Let (X, N, f) be a landscape where the neighborhood, N, is regular and symmetric. Then, f is elementary if and only if there exist two constants α and β such that:*

$$\operatorname*{avg}_{y \in N(x)}\{f(y)\} = \alpha f(x) + \beta \quad \forall x \in X \tag{5}$$

and the constants α and β are related to the offset b and the eigenvalue λ of f by the following expressions:

$$\alpha = 1 - \frac{\lambda}{d}, \qquad \beta = \frac{\lambda b}{d}. \tag{6}$$

Proof. If the landscape is elementary then Eqs. (5) and (6) follow from Theorem 1. Let us prove the reciprocal implication. We assume that (5) holds. Then, we can multiply both members by d to write:

$$\sum_{y \in N(x)} f(y) = d\,\alpha f(x) + d\,\beta = d\,f(x) + d\,(\alpha - 1) f(x) + d\,\beta.$$

If we subtract $d\,f(x)$ we have:

$$\sum_{y \in N(x)} f(y) - d\,f(x) = d\,(\alpha - 1) f(x) + d\,\beta.$$

At this point we must consider two cases. First, let us consider the case in which $\alpha = 1$, then we can write the previous equation in vectorial form as:

$$\Delta f = d\,\beta \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Multiplying by the row vector $(1, 1, \ldots, 1)$ in both members we get:

$$(1, 1, \ldots, 1)\Delta f = d\,\beta (1, 1, \ldots, 1) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = d\,\beta\,|X|.$$

However, due to the symmetry of the neighborhood it is possible to write:

$$d\,\beta\,|X| = ((1, 1, \ldots, 1)\Delta f)^T = f^T \Delta \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 0$$

which implies $\beta = 0$ since d and $|X|$ are greater than zero. Then, $-\Delta f = 0$ and f is an elementary landscape with $\lambda = 0$. This does not necessarily mean that f is a constant, since the neighborhood is not necessarily connected. If the neighborhood is connected, f must be a constant function.

Now, let us consider the case in which $\alpha \neq 1$. Then, we can write in vectorial form:

$$\Delta f = d\,(\alpha - 1) f + d\,\beta \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Taking into account the results of Proposition 1 and the definition of elementary landscape we can write:

$$-\Delta \left(f + \frac{\beta}{\alpha - 1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right) = -\Delta f = -d\,(\alpha - 1) \left(f + \frac{\beta}{\alpha - 1} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right)$$

and f is elementary with eigenvalue $\lambda = -d\,(\alpha - 1)$ and offset $b = -\beta/(\alpha - 1)$. □

The previous result provides a useful characterization of elementary landscapes that allows us to simplify the proof that a given landscape is elementary (or not). Although the result can be easily derived, to the best of our knowledge it has not been reported in the previous literature and it has not been used to check if a landscape is elementary. When f is not an elementary landscape equation (5) does not hold, but we can find a generalization of the equation that does hold if f is the sum of n elementary landscapes. This general expression is presented in the following

Theorem 2. *Let (X, N, f) be a landscape in which the neighborhood, N, is regular and symmetric. Then, f is the sum of n non-constant elementary landscapes f_i if and only if there exist some constants α_i for $i = 0, 1, \ldots, n$ such that*

$$\underset{y \in N(x)}{\mathrm{avg}}\{f(y)\} = \alpha_0 + \alpha_1 f(x) + \sum_{i=2}^{n} \alpha_i f_i(x) \quad \forall x \in X. \tag{7}$$

Proof. We can prove this by induction on n. In the base case, $n = 1$, Proposition 2 holds and the statement is true. For the inductive step let us assume that the statement is true for $n - 1$ and let us prove the result for n.

The function f is the sum of n elementary landscapes f_i, that is:

$$f = \sum_{i=1}^{n} f_i.$$

If we subtract f_n in the previous equality, then $f - f_n$ is the sum of $n-1$ elementary landscapes. We can apply the inductive hypothesis to compute the average value in the neighborhood of an arbitrary solution x. That is, a set of constants α_i exists such that:

$$\underset{y \in N(x)}{\mathrm{avg}}\{f(y) - f_n(y)\} = \alpha_0 + \alpha_1 (f(x) - f_n(x)) + \sum_{i=2}^{n-1} \alpha_i f_i(x). \tag{8}$$

Since f_n is an elementary landscape, according to Proposition 2 we can write

$$\operatorname*{avg}_{y\in N(x)}\{f_n(y)\} = \beta_0 + \beta_1 f_n(x)$$

and the previous expression can be written as:

$$\begin{aligned}\operatorname*{avg}_{y\in N(x)}\{f(y)\} &= \alpha_0 + \alpha_1(f(x) - f_n(x)) + \sum_{i=2}^{n-1}\alpha_i f_i(x) + \operatorname*{avg}_{y\in N(x)}\{f_n(y)\}\\ &= \alpha_0 + \alpha_1(f(x) - f_n(x)) + \sum_{i=2}^{n-1}\alpha_i f_i(x) + \beta_0 + \beta_1 f_n(x)\\ &= (\alpha_0 + \beta_0) + \alpha_1 f(x) + \sum_{i=2}^{n-1}\alpha_i f_i(x) + (\beta_1 - \alpha_1) f_n(x)\end{aligned}$$

and Eq. (7) holds for n.

Let us prove now the reciprocal implication. Let us assume that Eq. (7) holds for a given f, where all f_i are elementary landscapes under an appropriate neighborhood graph. Since f_n is a non-constant elementary landscape we can apply Proposition 2 and write $\operatorname{avg}\{f_n(y)\}_{y\in N(x)} = \beta_0 + \beta_1 f_n(x)$ with $\beta_1 \neq 0$. Then, Eq. (7) can be rewritten as:

$$\begin{aligned}\operatorname*{avg}_{y\in N(x)}\{f(y)\} &= \alpha_0 + \alpha_1 f(x) + \sum_{i=2}^{n-1}\alpha_i f_i(x) + \frac{\alpha_n}{\beta_1}\left(\operatorname*{avg}_{y\in N(x)}\{f_n(y)\} - \beta_0\right)\\ &= \alpha_0 + \alpha_1 f(x) + \sum_{i=2}^{n-1}\alpha_i f_i(x) + \operatorname*{avg}_{y\in N(x)}\{\alpha_n(f_n(y) - \beta_0)/\beta_1\}\,.\end{aligned}$$

In order to simplify the expressions let us define the function $g = \alpha_n(f_n - \beta_0)/\beta_1$. We can rewrite the previous expression in the following way:

$$\operatorname*{avg}_{y\in N(x)}\{f(y) - g(y)\} = \operatorname*{avg}_{y\in N(x)}\{f(y)\} - \operatorname*{avg}_{y\in N(x)}\{g(y)\} = \alpha_0 + \alpha_1 f(x) + \sum_{i=2}^{n-1}\alpha_i f_i(x).$$

Using the inductive hypothesis $f - g$ is the sum of $n-1$ elementary landscapes and this implies that f is the sum of n elementary landscapes since g is also an elementary landscape with the same eigenvalue as f_n. □

The previous result allows us to compute the average value of the objective function in the neighborhood of a given solution x from the value of the objective function f and $n-1$ elementary components f_i in x. This average value could be useful in practice for guiding a search method, but to access this information it requires that we know the elementary components of the objective function. We will highlight later that this decomposition is also useful in theory (for computing the autocorrelation coefficient). The next question is this: is it possible to write any objective function as the sum of elementary landscapes? The answer is affirmative when the neighborhood is symmetric, as the following theorem proves.

Theorem 3 (*Elementary Landscape Decomposition*). *Let (X, N, f) be a landscape where the neighborhood, N, is symmetric. Then, there exist n elementary landscapes with $1 \leq n \leq |X|$ such that f can be written as the sum of all of these n elementary landscapes.*

Proof. From linear algebra we know that if a square real matrix $-\Delta$ of size $|X|$ is symmetric then there exists an orthogonal basis of the vectorial space $\mathbb{R}^{|X|}$ that is composed of eigenvectors of $-\Delta$. Then, we can write every vector of $\mathbb{R}^{|X|}$ as the weighted sum of the vectors in the orthogonal basis. If we translate these concepts into the landscapes language this means that for any symmetric neighborhood N it is possible to find an orthogonal basis composed of elementary functions. Then, any function f can be written as the sum of a set of elementary landscapes. □

3. Frequency assignment problem (FAP)

The frequency assignment problem is the last step in the layout of a radio network, like a 2G (second generation) cellular mobile network. Prior to tackling this problem, the network designer has to address some other issues: where to install the base stations or how to set configuration parameters of the antennae (tilt, azimuth, etc.), among others [9]. Once the sites for the base stations are selected and the sector layout is decided, the number of transceivers (TRXs) to be installed per sector has to be fixed. This number depends on the traffic demand that the corresponding sector is expected to support. Frequency assignment lies on the assignment of a channel (a frequency band) to every TRX [10]. The optimization problem arises because the usable radio spectrum is generally very scarce and, consequently, channels have to be reused by many TRXs in the network.

However, the multiple use of the same channel may cause interference that might reduce the quality of service down to unsatisfactory levels. Indeed, significant interference may occur if the same or adjacent-channels are used in neighboring overlapping cells. Computing this level of interference is a difficult task, which depends not only on the channels, but also on the radio signals and the properties of the environment. Several ways of quantifying this interference exist, ranging from theoretical methods to extensive measurements [11]. They all result in the so-called *interference matrix*, denoted by M. Each element m_{ij} of M indicates the degradation of the network quality if TRXs i and j operate on the same channel. This is called *co-channel interference*. In addition to co-channel interference there may also be a so-called *adjacent-channel interference*, which occurs when two TRXs operate on adjacent channels (i.e., one TRX operates on channel p and the other on channel $p+1$ or $p-1$). Co-channel and adjacent-channel interference are the most important ones in the design of a radio network. But we could also be interested in considering interference due to overlapping of channels with a larger separation. This is in accordance with real-world applications, since the amount of interference between two channels depends on the separation of the channels [12].

Thus, in our generalized form of the frequency assignment problem, we consider both co-channel interference and adjacent channel interference as well as interference due to frequencies with a larger separation. We can then generate specialized versions of FAP. For example, by setting the adjacent channel interference to zero, the basic form of the FAP is created, in which only co-channel interference is considered.

We can assign a cost to each possible interference that can occur in a channel assignment. Then, the objective in FAP is to minimize the cost due to interference in a radio network. An additional generalization of this problem also considers the possibility of additional costs due to the mere fact that a given channel is used by a given TRX, e.g., a fee could be charged to a telecommunication company for using a channel in a given location. In general, the set of channels that can be assigned to each TRX might be different. We assume that the valid channels of each TRX are sorted and we use an integer number to represent their position in the sorted set. We also assume, without loss of generality, that the number of valid channels is the same in all the TRXs, and we denote this number by r. We denote as n the number of TRXs in the radio network.

In order to take into account all the previous considerations and keep a compact formulation of the problem we define an array of weights $w \in \mathbb{R}^{n\times n\times r\times r}$ in which we denote each element with $w_{i,j}^{p,q}$ where $i, j \in \{1, \dots, n\}$ and $p, q \in \{1, \dots, r\}$. We can interpret the element $w_{i,j}^{p,q}$ as the cost of having channel p in TRX i and channel q in TRX j. Before giving the expression for the cost function, let us define the solution space X. One solution for this problem is a map from the set of TRXs, denoted with V, to the set of possible channels $F = \{1, 2, \dots, r\}$. Thus, the solution space is $X = F^V$. Using the array of weights we can define the cost function as:

$$f(x) = \sum_{i,j=1}^{n} w_{i,j}^{x(i),x(j)}. \tag{9}$$

The cost element $w_{j,i}^{q,p}$ has the same meaning as $w_{i,j}^{p,q}$, so we can set one of them to zero. However, for the sake of clarity and without loss of generality, we will take the convention that $w_{i,j}^{p,q} = w_{j,i}^{q,p}$ for all i, j, p, q. Then, the cost element $w_{i,j}^{p,q}$ with $i \neq j$ must be interpreted as half the cost of having channel p in TRX i and channel q in TRX j. If $i = j$ the element $w_{i,i}^{p,p}$ is the additional cost of having channel p in TRX i.

4. Landscape decomposition of the generalized FAP

Once we have defined the solution space X and the cost function f, we need to specify the neighborhood N for the problem. We say that two solutions $x, y \in X$ are neighbors if there exists a transceiver $v \in V$ such that $x(v) \neq y(v)$ and for the remaining transceivers w we have $x(w) = y(w)$. This neighborhood is symmetric, connected, and regular with size $d = n(r-1)$.

Once we have completely defined the landscape, let us rewrite the cost function (9) in a more convenient way, as a linear combination of data-independent functions:

$$f(x) = \sum_{i,j=1}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} \varphi_{i,j}^{p,q}(x) \tag{10}$$

where $\varphi_{i,j}^{p,q}$ is an indicator function defined as:

$$\varphi_{i,j}^{p,q}(x) = \begin{cases} 1 & \text{if } x(i) = p \wedge x(j) = q \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Now we can focus on the indicator functions $\varphi_{i,j}^{p,q}$. In the following we prove that these functions are not always elementary in the considered neighborhood, but each one is the sum of at most two elementary landscapes. Furthermore, the eigenvalues of the two elementary landscapes are independent of the values of i, j, p and q. In order to prove this statement we need to introduce an auxiliary function (or family of functions) $\phi_{i,j,\alpha}^{p,q}$ defined as:

$$\phi_{i,j,\alpha}^{p,q}(x) = \begin{cases} \alpha & \text{if } x(i) = p \wedge x(j) = q \\ -1 & \text{if } x(i) = p \oplus x(j) = q \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

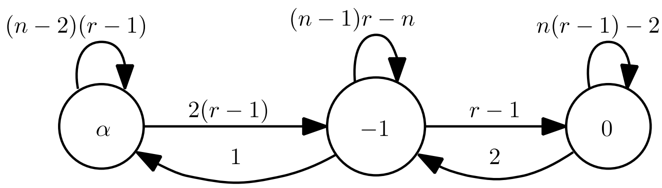

Fig. 1. Transition graph for functions ϕ_α.

where $\oplus$ denotes the exclusive-or logic operator. That is, $\phi^{p,q}_{i,j,\alpha}(x)$ is α if the transceivers i and j operate on their target channels p and q, respectively, -1 if only one (not both) of the transceivers operates on its target channel, and 0 if none of the transceivers operates on their target channels. In the following, for the sake of simplicity we will remove the parameters i, j, p, and q from the name of the function when there is no confusion. For this function the following result holds:

Lemma 2. *For the neighborhood N defined above and given i, j, p and q with $i \neq j$, the function $\phi^{p,q}_{i,j,\alpha}$ is an elementary landscape with eigenvalue $\lambda = 2r$ for $\alpha = r-2$ and with $\lambda = r$ for $\alpha = -2$.*

Proof. For the proof we use the characterization of elementary functions given in Proposition 2. We distinguish three different cases which are symbolically represented in Fig. 1. In the figure, each node represents the set of solutions for which one of the three branches in (12) is true. We label the nodes with the value that $\phi^{p,q}_{i,j,\alpha}$ takes for all the solutions in that node. There exists an arc (u, v) if all the solutions in node u have at least one neighboring solution in node v. The label of arc (u, v) is the number of neighbors in v that any solution in u has.

Given a solution x, the average in the neighborhood of x can be easily computed taking into account the condition that x fulfills:

- Case $\phi_\alpha(x) = \alpha$. In this case the transceivers i and j operate on their target channels. If we focus on the neighboring solutions we find that there are $2(r-1)$ neighbors with $\phi_\alpha(y) = -1$ and no neighbor with $\phi_\alpha(y) = 0$. Then, the average value of $\phi_\alpha(y)$ in the neighborhood is
$$\underset{y\in N(x)}{\mathrm{avg}}\{\phi_\alpha(y)\} = \frac{(d-2(r-1))\alpha - 2(r-1)}{d} = \alpha - \frac{2(r-1)(\alpha+1)}{d}.$$
- Case $\phi_\alpha(x) = -1$. In this case one of the transceivers, either i or j, operates on its target channel. There is one neighbor with $\phi_\alpha(y) = \alpha$ and $(r-1)$ neighbors with $\phi_\alpha(y) = 0$. Then, the average value of $\phi_\alpha(y)$ in the neighborhood is
$$\underset{y\in N(x)}{\mathrm{avg}}\{\phi_\alpha(y)\} = \frac{(d-r)(-1)+\alpha}{d} = -1 + \frac{r+\alpha}{d}.$$
- Case $\phi_\alpha(x) = 0$. In this case none of the transceivers operates on their target channels. There are two neighbors with $\phi_\alpha(y) = -1$. Then, the average value of $\phi_\alpha(y)$ in the neighborhood is
$$\underset{y\in N(x)}{\mathrm{avg}}\{\phi_\alpha(y)\} = \frac{-2}{d}.$$

Once we have computed the average value of $\phi_\alpha(y)$ in the neighborhood for the three cases, we need to solve the following linear equation system in order to check whether or not it can be solvable and Proposition 2 can be applied:
$$\begin{pmatrix} \alpha & 1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha - \frac{2(r-1)(\alpha+1)}{d} \\ -1 + \frac{r+\alpha}{d} \\ -\frac{2}{d} \end{pmatrix}.$$

The previous system has three equations and two variables, so it could be unsolvable. However, the system can be solved for two values of α, namely: $\alpha = r-2$ and $\alpha = -2$ with solution $a = 1 - \frac{r+\alpha+2}{d}$ and $b = -2/d$. This means that we can write:
$$\underset{y\in N(x)}{\mathrm{avg}}\{\phi_\alpha(y)\} = \phi_\alpha(x)\left(1 - \frac{r+\alpha+2}{d}\right) - \frac{2}{d}.$$

According to Proposition 2, from the previous expression we conclude that the $\phi_\alpha(x)$ functions are elementary landscapes with eigenvalue $\lambda = r+\alpha+2$ and average value (offset) $\bar{\phi}_\alpha = -2/(r+\alpha+2)$. For $\alpha = r-2$ we have eigenvalue $\lambda = 2r$ and $\bar{\phi}_{r-2} = -1/r$, while for $\alpha = -2$ we have eigenvalue $\lambda = r$ and $\bar{\phi}_{-2} = -2/r$. $\square$

Using the auxiliary functions, the indicator function $\varphi^{p,q}_{i,j}$ can be written as:
$$\varphi^{p,q}_{i,j} = \frac{1}{r}\left(\phi^{p,q}_{i,j,r-2} - \phi^{p,q}_{i,j,-2}\right) \tag{13}$$
independently of the values of the parameters i, j, p and q. With this expression and the result of Lemma 2 we can state that if $i \neq j$ then $\varphi^{p,q}_{i,j}$ is the sum of two elementary landscapes whose eigenvalues are independent of the parameters. If $i = j$ we cannot apply Lemma 2 but we have the following result.

Lemma 3. *For the neighborhood N defined above, the function $\varphi_{i,i}^{p,p}$ is an elementary landscape with eigenvalue $\lambda = r$ and $\varphi_{i,i}^{p,q} = 0$ for $p \neq q$.*

Proof. The function $\varphi_{i,i}^{p,q}$ can be written using the Kronecker delta as $\varphi_{i,i}^{p,q}(x) = \delta_{x(i)}^{p}\delta_{x(i)}^{q}$. According to the properties of δ, $\varphi_{i,i}^{p,q} = 0$ if $p \neq q$. If $p = q$ then we have $\varphi_{i,i}^{p,p}(x) = \delta_{x(i)}^{p}$. For this case we distinguish two cases:

- Case $\varphi_{i,i}^{p,p}(x) = 1$. In this case the transceiver i operates on channel p. There are $r - 1$ neighbors with value 0 and the remaining ones have value 1. Then, the average value of $\varphi_{i,i}^{p,p}(y)$ in the neighborhood is

$$\operatorname*{avg}_{y\in N(x)}\{\varphi_{i,i}^{p,p}(y)\} = \frac{d-(r-1)}{d} = 1 - \frac{r}{d} + \frac{1}{d}.$$

- Case $\varphi_{i,i}^{p,p}(x) = 0$. In this case the transceiver i does not operate on channel p. There is one neighbor with value 1 and the remaining ones have value 0. Then, the average value of $\varphi_{i,i}^{p,p}(y)$ in the neighborhood is

$$\operatorname*{avg}_{y\in N(x)}\{\varphi_{i,i}^{p,p}(y)\} = \frac{1}{d}.$$

Once we have computed the average value of $\varphi_{i,i}^{p,p}(y)$ in the neighborhood for the two cases, we need to solve the linear equation system:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 - \frac{r}{d} + \frac{1}{d} \\ \frac{1}{d} \end{pmatrix}.$$

The previous system has two equations and two variables, and the solution is $a = 1 - \frac{r}{d}$ and $b = 1/d$. This means that we can write:

$$\operatorname*{avg}_{y\in N(x)}\{\varphi_{i,i}^{p,p}(y)\} = \varphi_{i,i}^{p,p}(x)\left(1 - \frac{r}{d}\right) + \frac{1}{d}.$$

From the previous expression we conclude that all the $\varphi_{i,i}^{p,p}(x)$ functions are elementary landscapes with eigenvalue $\lambda = r$ and average value $\bar{\varphi}_{i,i}^{p,p} = 1/r$. □

As a consequence of the two previous lemmas, we can state the first main contribution of this paper, which is the following

Theorem 4. *For the neighborhood N defined above for the generalized FAP, the function f is the sum of two elementary landscapes with eigenvalues $\lambda_1 = 2r$ and $\lambda_2 = r$.*

Proof. According to (10) we can write:

$$\begin{aligned} f &= \sum_{i,j=1}^{n}\sum_{p,q=1}^{r} w_{i,j}^{p,q}\varphi_{i,j}^{p,q} \\ &= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w_{i,j}^{p,q}\left(\phi_{i,j,r-2}^{p,q} - \phi_{i,j,-2}^{p,q}\right) + \sum_{i=1}^{n}\sum_{p=1}^{r} w_{i,i}^{p,p}\varphi_{i,i}^{p,p} \\ &= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w_{i,j}^{p,q}\phi_{i,j,r-2}^{p,q} - \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w_{i,j}^{p,q}\phi_{i,j,-2}^{p,q} + \sum_{i=1}^{n}\sum_{p=1}^{r} w_{i,i}^{p,p}\varphi_{i,i}^{p,p}. \end{aligned}$$

Now we combine the terms of the previous expression in the following way and we define two new functions:

$$f_{2r} = \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w_{i,j}^{p,q}\phi_{i,j,r-2}^{p,q} \tag{14}$$

$$f_r = -\frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w_{i,j}^{p,q}\phi_{i,j,-2}^{p,q} + \sum_{i=1}^{n}\sum_{p=1}^{r} w_{i,i}^{p,p}\varphi_{i,i}^{p,p}. \tag{15}$$

The function f_{2r} is elementary with $\lambda = 2r$ since it is the weighted sum of elementary landscapes with $\lambda = 2r$. For the same reason, f_r is elementary with $\lambda = r$. Then, $f = f_{2r} + f_r$ is the sum of two elementary landscapes with eigenvalues $\lambda_1 = 2r$ and $\lambda_2 = r$. □

Since f is not an elementary landscape, Proposition 2 does not hold for f, that is, we cannot compute the average function value in the neighborhood of one solution $\text{avg}\{f(y)\}_{y\in N(x)}$ using the function value of the solution $f(x)$. Instead, according to Theorem 2 we also need the value of one of the elementary components for the solution as the following proposition holds.

Proposition 3. *For the neighborhood N defined above and the cost function f of the generalized FAP the following expressions both hold:*

$$\underset{y\in N(x)}{\text{avg}}\{f(y)\} = \left(1-\frac{2r}{d}\right)f(x) + \frac{r}{d}f_r(x) + \frac{r}{d}(2\bar{f} - \bar{f}_r) \tag{16}$$

$$\underset{y\in N(x)}{\text{avg}}\{f(y)\} = \left(1-\frac{r}{d}\right)f(x) - \frac{r}{d}f_{2r}(x) + \frac{r}{d}(\bar{f} + \bar{f}_{2r}). \tag{17}$$

Proof. Since $f = f_{2r} + f_r$ we can write:

$$\begin{aligned}
\underset{y\in N(x)}{\text{avg}}\{f(y)\} &= \underset{y\in N(x)}{\text{avg}}\{f_{2r}(y)\} + \underset{y\in N(x)}{\text{avg}}\{f_r(y)\} \\
&= f_{2r}(x) + \frac{2r}{d}\left(\bar{f}_{2r} - f_{2r}(x)\right) + f_r(x) + \frac{r}{d}\left(\bar{f}_r - f_r(x)\right) \\
&= f(x) - \frac{2r\, f_{2r}(x) + r\, f_r(x)}{d} + \frac{2r\,\bar{f}_{2r} + r\,\bar{f}_r}{d}.
\end{aligned} \tag{18}$$

Now we can advance in two directions. If we want to obtain (16) we replace f_{2r} by $f - f_r$ in (18) and we write:

$$\begin{aligned}
\underset{y\in N(x)}{\text{avg}}\{f(y)\} &= f(x) - \frac{2r\, f_{2r}(x) + r\, f_r(x)}{d} + \frac{2r\,\bar{f}_{2r} + r\,\bar{f}_r}{d} \\
&= f(x) - \frac{2r\, f(x) - r\, f_r(x)}{d} + \frac{2r\,\bar{f} - r\,\bar{f}_r}{d} \\
&= \left(1-\frac{2r}{d}\right)f(x) + \frac{r}{d}f_r(x) + \frac{r}{d}(2\bar{f} - \bar{f}_r).
\end{aligned}$$

If we want to obtain (17) we replace f_r by $f - f_{2r}$ in (18) and we write:

$$\begin{aligned}
\underset{y\in N(x)}{\text{avg}}\{f(y)\} &= f(x) - \frac{2r\, f_{2r}(x) + r\, f_r(x)}{d} + \frac{2r\bar{f}_{2r} + r\,\bar{f}_r}{d} \\
&= f(x) - \frac{r\, f(x) + r\, f_{2r}(x)}{d} + \frac{r\,\bar{f} + r\,\bar{f}_{2r}}{d} \\
&= \left(1-\frac{r}{d}\right)f(x) - \frac{r}{d}f_{2r}(x) + \frac{r}{d}(\bar{f} + \bar{f}_{2r}). \quad \square
\end{aligned}$$

5. Elementary landscapes in the FAP

In this section we study under what circumstances a specific instance of the generalized FAP corresponds to a single elementary landscape. We want to derive some conditions that can later be checked on actual problem data to determine whether a given instance of the problem is elementary or not. As a consequence, we could then define problem subclasses for the generalized FAP that are also elementary.

As we saw in the previous section, the function f we are considering is, in the general case, the sum of two elementary functions, f_{2r} and f_r, defined according to Eqs. (14) and (15), respectively. The data of the problem is the array of weights $w_{i,j}^{p,q}$, which can be considered a four-rank tensor. Our purpose now is to investigate what are the conditions that this array of weights must fulfill for the function f to be an elementary landscape. Since we know the decomposition of f into elementary components we just have to study under what conditions f_{2r} and f_r are constant functions. In effect, if f_{2r} is a constant function, then $f = f_{2r} + f_r$ will be an elementary landscape with $\lambda = r$. In the same way, if f_r is a constant, then f will be an elementary landscape with $\lambda = 2r$. Thus, our first goal is to investigate what are the conditions that the weights $w_{i,j}^{p,q}$ must fulfill in order for the elementary components of f to be a constant. This is the objective of Lemmas 4 and 5. Later, in Theorem 5 we summarize the results.

To simplify the notation in the following results, let us define the following matrix based on the weight array w:

$$u_{i,a} = \sum_{\substack{j=1\\j\neq i}}^{n}\sum_{q=1}^{r} w_{i,j}^{a,q} + \frac{r}{2}w_{i,i}^{a,a} = \sum_{\substack{j=1\\j\neq i}}^{n}\sum_{q=1}^{r} w_{j,i}^{q,a} + \frac{r}{2}w_{i,i}^{a,a}. \tag{19}$$

Let us start with f_r. The following lemma presents the conditions under which f_r is a constant function.

Lemma 4. *Given the neighborhood N defined for the FAP, the elementary function f_r defined in Eq. (15) and the symmetry requirement of the array of weights $w_{i,j}^{p,q} = w_{j,i}^{q,p}$, then f_r is a constant function if and only if the weights satisfy the following condition: for all $i \in V$ there exists a constant c_i such that*

$$\forall a \in F \quad u_{i,a} = c_i. \tag{20}$$

Proof. According to Eq. (12) we can write $\phi_{i,j,-2}^{p,q}(x) = -(\delta_{x(i)}^p + \delta_{x(j)}^q)$, where $i \neq j$. If $i = j$ then we have $\varphi_{i,i}^{p,p}(x) = \delta_{x(i)}^p$. Then, we can write f_r as:

$$\begin{aligned} f_r(x) &= -\frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w_{i,j}^{p,q}\phi_{i,j,-2}^{p,q}(x) + \sum_{i=1}^{n}\sum_{p=1}^{r} w_{i,i}^{p,p}\varphi_{i,i}^{p,p}(x) \\ &= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w_{i,j}^{p,q}(\delta_{x(i)}^p + \delta_{x(j)}^q) + \sum_{i=1}^{n}\sum_{p=1}^{r} w_{i,i}^{p,p}\delta_{x(i)}^p \\ &= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\sum_{p,q=1}^{r} w_{i,j}^{p,q}\delta_{x(i)}^p + \sum_{p,q=1}^{r} w_{i,j}^{p,q}\delta_{x(j)}^q\right) + \sum_{i=1}^{n} w_{i,i}^{x(i),x(i)} \\ &= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\sum_{q=1}^{r} w_{i,j}^{x(i),q} + \sum_{p=1}^{r} w_{i,j}^{p,x(j)}\right) + \sum_{i=1}^{n} w_{i,i}^{x(i),x(i)}. \end{aligned}$$

If we rename the mute variable q by p and use the property of the weights array $w_{i,j}^{p,q} = w_{j,i}^{q,p}$ we can rewrite $f_r(x)$ as

$$\begin{aligned} f_r(x) &= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\sum_{p=1}^{r} w_{i,j}^{x(i),p} + \sum_{p=1}^{r} w_{j,i}^{x(j),p}\right) + \sum_{i=1}^{n} w_{i,i}^{x(i),x(i)} \\ &= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p=1}^{r} w_{i,j}^{x(i),p} + \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p=1}^{r} w_{j,i}^{x(j),p} + \sum_{i=1}^{n} w_{i,i}^{x(i),x(i)}. \end{aligned}$$

We should notice that the first and second terms in the last expression are the same (the only difference are mute variables i and j). Then, we can write:

$$f_r(x) = \frac{2}{r}\sum_{i=1}^{n}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{p=1}^{r} w_{i,j}^{x(i),p} + \frac{r}{2}w_{i,i}^{x(i),x(i)}\right) = \frac{2}{r}\sum_{i=1}^{n} u_{i,x(i)}.$$

In order for the previous function to be a constant it is necessary that for all i there exists a constant c_i such that $u_{i,x(i)} = c_i$ for all the possible values of $x(i)$. To prove this, let us suppose that this is not true, that is, there exists a transceiver i and two solutions x and y such that $u_{i,x(i)} \neq u_{i,y(i)}$. Let us also assume that solutions x and y only differ in the channel of TRX i. Then, the following expression is true: $u_{i,x(k)} = u_{i,y(k)}$ for all $k \neq i$. Thus, $\sum_{i=1}^{n} u_{i,x(i)} \neq \sum_{i=1}^{n} u_{i,y(i)}$, and the function $f_r(x) \neq f_r(y)$. Since all the possible values of $x(i)$ are the elements of F, the condition for f_r to be a constant is that for each i there exists a constant c_i such that $u_{i,a} = c_i$ for all $a \in F$. And this is Eq. (20).

This condition is also sufficient, since if for all i there exists a constant c_i such that $u_{i,a} = c_i$ for all $a \in F$, then

$$f_r(x) = \frac{2}{r}\sum_{i=1}^{n} u_{i,x(i)} = \frac{2}{r}\sum_{i=1}^{n} c_i.$$

The value $f_r(x)$ is then independent of x and f_r is a constant function. □

Let us now characterize when f_{2r} is constant. In this case, we have the following result.

Lemma 5. *Given the neighborhood N defined for the FAP, the elementary function f_{2r} defined in Eq. (14) and the symmetry requirement of the array of weights $w_{i,j}^{p,q} = w_{j,i}^{q,p}$, then f_{2r} is a constant function if and only if for each pair of transceivers $i, j \in V$ there exist two vectors $\varpi_{(i,j)}$ and $\pi_{(i,j)}$ such that it is possible to write:*

$$w_{i,j}^{p,q} = \varpi_{(i,j),p} + \pi_{(i,j),q}. \tag{21}$$

Proof. According to Eq. (12) we can write $\phi^{p,q}_{i,j,r-2}(x) = r\delta^{p}_{x(i)}\delta^{q}_{x(j)} - \delta^{p}_{x(i)} - \delta^{q}_{x(j)}$, using the Kronecker delta. Then, we can write f_{2r} as:

$$
\begin{aligned}
f_{2r}(x) &= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w^{p,q}_{i,j}\phi^{p,q}_{i,j,r-2} \\
&= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{p,q=1}^{r} w^{p,q}_{i,j}(r\delta^{p}_{x(i)}\delta^{q}_{x(j)} - \delta^{p}_{x(i)} - \delta^{q}_{x(j)}) \\
&= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\sum_{p,q=1}^{r} r\, w^{p,q}_{i,j}\delta^{p}_{x(i)}\delta^{q}_{x(j)} - \sum_{p,q=1}^{r} w^{p,q}_{i,j}\delta^{p}_{x(i)} - \sum_{p,q=1}^{r} w^{p,q}_{i,j}\delta^{q}_{x(j)}\right) \\
&= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(r\, w^{x(i),x(j)}_{i,j} - \sum_{q=1}^{r} w^{x(i),q}_{i,j} - \sum_{p=1}^{r} w^{p,x(j)}_{i,j}\right) \\
&= \frac{1}{r}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(r\, w^{x(i),x(j)}_{i,j} - 2\sum_{q=1}^{r} w^{x(i),q}_{i,j}\right) \\
&= \sum_{\substack{i,j=1\\ i\neq j}}^{n} w^{x(i),x(j)}_{i,j} - \frac{2}{r}\sum_{i=1}^{n}\left(u_{i,x(i)} - \frac{r}{2}w^{x(i),x(i)}_{i,i}\right).
\end{aligned}
$$

If f_{2r} is constant, then given two solutions x and y that differ only in the channel of an arbitrary $l \in V$, it must hold that $f_{2r}(x) = f_{2r}(y)$. The contrary is also true, that is, if for all $l \in V$ and for all the solutions x and y that only differ in the channel of TRX l it happens that $f_{2r}(x) = f_{2r}(y)$ then f_{2r} is a constant function. To prove this let us suppose that we have two arbitrary solutions x and y. Then, we can define a series of solutions $x = s_1, s_2, \ldots, s_{n+1} = y$ in which two consecutive solutions only differ in the channel of one transceiver, that is, s_i and s_{i+1} differ in the channel of transceiver i and $s_i(i) = x(i)$ and $s_{i+1}(i) = y(i)$. Since $f_{2r}(s_i) = f_{2r}(s_{i+1})$, then $f_{2r}(x) = f_{2r}(y)$.

Then, we focus now on all the pairs of solutions x and y that differ in the channel of an arbitrary transceiver l. For f_{2r} to be a constant function the value of $f_{2r}(x) - f_{2r}(y)$ must be zero. To simplify the notation let us call $a = x(l)$ and $b = y(l)$. Then, we can write:

$$
\begin{aligned}
f_{2r}(x) - f_{2r}(y) &= \sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(w^{x(i),x(j)}_{i,j} - w^{y(i),y(j)}_{i,j}\right) - \frac{2}{r}\sum_{i=1}^{n}\left(u_{i,x(i)} - u_{i,y(i)} - \frac{r}{2}w^{x(i),x(i)}_{i,i} + \frac{r}{2}w^{y(i),y(i)}_{i,i}\right) \\
&= \sum_{\substack{j=1\\ j\neq l}}^{n}\left(w^{a,x(j)}_{l,j} - w^{b,x(j)}_{l,j}\right) + \sum_{\substack{i=1\\ i\neq l}}^{n}\left(w^{x(i),a}_{i,l} - w^{x(i),b}_{i,l}\right) - \frac{2}{r}\left(u_{l,a} - u_{l,b} - \frac{r}{2}w^{a,a}_{l,l} + \frac{r}{2}w^{b,b}_{l,l}\right) \\
&= 2\sum_{\substack{j=1\\ j\neq l}}^{n}\left(w^{a,x(j)}_{l,j} - w^{b,x(j)}_{l,j}\right) - \frac{2}{r}\left(u_{l,a} - u_{l,b} - \frac{r}{2}w^{a,a}_{l,l} + \frac{r}{2}w^{b,b}_{l,l}\right).
\end{aligned}
$$

Taking into account that $f_{2r}(x) - f_{2r}(y)$ must be 0 we can write:

$$
2\sum_{\substack{j=1\\ j\neq l}}^{n}\left(w^{a,x(j)}_{l,j} - w^{b,x(j)}_{l,j}\right) = \frac{2}{r}\left(\frac{r}{2}w^{b,b}_{l,l} - \frac{r}{2}w^{a,a}_{l,l} + u_{l,a} - u_{l,b}\right). \tag{22}
$$

The previous expression must be true for all the solutions x. In particular, it must be true for all the values that $x(j)$ can take. This implies that for all $j \in V$ there must exist a constant $c^{a,b}_{l,j}$ such that for all $q \in F$ the following must hold: $w^{a,q}_{l,j} - w^{b,q}_{l,j} = c^{a,b}_{l,j}$. If we focus on matrix $w_{l,j}$, the previous expression means that rows a and b of that matrix must differ in a constant. Since the expression must be valid for all l, j, a and b, all the rows in each matrix $w_{l,j}$ must differ in a constant (which depends on the values of l, j and the rows, in general). It is not difficult to see that if the rows of matrix $w_{l,j}$ differ in a constant then there exist two vectors, which we denote with $\varpi_{(l,j)}$ and $\pi_{(l,j)}$, such that $w^{p,q}_{l,j} = \varpi_{(l,j),p} + \pi_{(l,j),q}$. This way we have proven that condition (21) is necessary.

Now, let us prove that condition (21) is also sufficient. Let us insert the definition of $w_{l,j}$ based on the vectors $\varpi_{(l,j)}$ and $\pi_{(l,j)}$ in (22). First, we must observe that the constant $c^{a,b}_{l,j} = w^{a,q}_{l,j} - w^{b,q}_{l,j}$ can be written as:

$$
c^{a,b}_{l,j} = \varpi_{(l,j),a} + \pi_{(l,j),q} - \varpi_{(l,j),b} - \pi_{(l,j),q} = \varpi_{(l,j),a} - \varpi_{(l,j),b}.
$$

Then, (22) can be written as:

$$2\sum_{\substack{j=1\\j\neq l}}^{n}\left(\varpi_{(l,j),a}-\varpi_{(l,j),b}\right)=\frac{2}{r}\sum_{\substack{j=1\\j\neq l}}^{n}\sum_{q=1}^{r}\left(w_{l,j}^{a,q}-w_{l,j}^{b,q}\right)$$
$$=\frac{2}{r}\sum_{\substack{j=1\\j\neq l}}^{n}\sum_{q=1}^{r}\left(\varpi_{(l,j),a}-\varpi_{(l,j),b}\right).$$

Since both terms are equal, condition (22) is true and f_{2r} is constant. □

Using Lemmas 4 and 5 we can summarize the results in the following

Theorem 5. *Given the neighborhood N defined for the generalized FAP and the function f defined in Eq. (10), according to the decomposition of f in elementary components, four scenarios are possible depending on the array of weights w:*

- *f is a constant function, that is, an elementary landscape with $\lambda = 0$, if conditions (20) and (21) hold.*
- *f is a non-constant elementary landscape defined in (14) with $\lambda = 2r$ if condition (20) holds but condition (21) does not hold.*
- *f is a non-constant elementary landscape defined in (15) with $\lambda = r$ if condition (21) holds but condition (20) does not hold.*
- *f is a sum of two non-constant elementary landscapes f_{2r} and f_r defined in (14) and (15), respectively, if conditions (21) and (20) do not hold.*

Proof. From Lemmas 4 and 5 the three first cases are direct. To complete the proof we need to discard the possibility that in the fourth case f is constant. Let us suppose that f is constant in this case. Then, $f_{2r} = f - f_r$ should be at the same time an elementary landscape with $\lambda = r$ and $\lambda = 2r$, but this is not possible since $r > 0$. Then, f cannot be constant in the fourth case. □

In the following we will study some subclasses of the problem that are elementary with $\lambda = 2r$. Condition (20) is the most general one for this case, but we can obtain a simpler condition that implies the previous one. The simpler condition is that for all $i, j \in V$ with $i \neq j$ there exists a constant $c_{i,j}$ such that it holds:

$$\forall a \in F \quad \sum_{q=1}^{r} w_{i,j}^{a,q} = c_{i,j} \tag{23}$$

and $w_{i,i}^{a,a} = 0$ for all $i \in V$ and all $a \in F$. In other words, for each matrix $w_{i,j}$ the rows must sum to the same value. Since $w_{i,j} = (w_{j,i})^T$, the columns must also sum to the same value.

All the instances of the general FAP that fulfill condition (23) are elementary with constant $\lambda = 2r$. We highlight three important subproblems that satisfy (23):

- **Symmetric Adjacent FAP**: In this case, two transceivers have a contribution to the cost function if their channels are the same (direct cost) or adjacent (adjacent cost). Frequencies 1 and r are considered adjacent. The direct cost and the adjacent cost do not depend on the channels but they can depend on the transceivers. For each pair of transceivers $i, j \in V$ the matrix $w_{i,j}$ is of the form:

$$w_{i,j} = \begin{pmatrix} d & a & 0 & \cdots & 0 & a \\ a & d & a & \cdots & 0 & 0 \\ 0 & a & d & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & 0 & 0 & \cdots & a & d \end{pmatrix}.$$

- **Basic FAP (weighted graph coloring)**: In this case, only direct costs are considered, that is, two transceivers have a contribution to the cost function if their channels are the same. The direct cost can be different for each pair of transceivers. For each pair of transceivers $i, j \in V$ the matrix $w_{i,j}$ is of the form:

$$w_{i,j} = \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & d & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d \end{pmatrix}.$$

  This subproblem is a kind of graph coloring in which each edge has a weight that can be different from 1.
- **Graph coloring**: This is a particular case of the previous situation in which the direct cost is 1 or 0. Thus, the matrix $w_{i,j}$ is the identity or the zero matrix (depending on the pair of edges).

To finish this section let us summarize the classification of subproblems of FAP according to the landscape decomposition. The two main subproblems with practical interest are the so-called *Basic Frequency Assignment Problem* (BFAP) and the

Asymmetric Adjacent Frequency Assignment Problem (AAFAP). The BFAP is defined to be exactly the same as the Weighted Graph Coloring problem. Conflicts may occur when two different transceivers are assigned the same channel. This subproblem is an elementary landscape with constant $\lambda = 2r$, as we have just seen. The AAFAP, on the other hand, can be decomposed as the sum of two elementary landscapes but, in the general case, it is not an elementary landscape. A third subproblem of FAP with no practical application but with a theoretical interest is the *Symmetric Adjacent Frequency Assignment Problem* (SAFAP), which, as we have just shown, is an elementary landscape. Finally, all the subproblems of FAP in which conflicts between channels with a separation larger than 1 are considered can be decomposed as a sum of two elementary landscapes, and they are not always elementary.

6. Autocorrelation coefficient of FAP

We are going now to use the elementary landscape decomposition found in the previous section to derive a closed form formula for the autocorrelation coefficient of the FAP. The *autocorrelation coefficient* ξ of a problem is a parameter proposed by Angel and Zissimopoulos [13] that gives a measure of its ruggedness, which is related to the performance that local search methods have on the problem. In [14] the same authors provide a set of experimental results using a Simulated Annealing algorithm that suggests that the higher the value of ξ the better the performance of the algorithm. One possible explanation for this observed behavior is given by the *autocorrelation length conjecture*, which claims that in many landscapes the number of local optima is directly related to its ruggedness [15].

The definition of ξ is based on the autocorrelation function proposed by Weinberger [16]. Let us consider a random walk $\{x_0, x_1, \ldots\}$ on the solution space such that $x_{i+1} \in N(x_i)$. The autocorrelation function r is defined as:

$$r(s) = \frac{\langle f(x_t) f(x_{t+s}) \rangle_{x_0,t} - \langle f(x_t) \rangle^2_{x_0,t}}{\langle f(x_t)^2 \rangle_{x_0,t} - \langle f(x_t) \rangle^2_{x_0,t}} \tag{24}$$

where the averages, denoted with $\langle \cdot \rangle$ in this case, are computed over all the starting solutions x_0 and all the solutions in the sequence. The autocorrelation coefficient is then defined as $\xi = \frac{1}{1-r(1)}$.

Stadler [3] proved that if $f = \sum_i a_i \phi_i$ is a Fourier expansion of f in a landscape, then the autocorrelation function of f is given by

$$r(s) = \sum_{i \neq 0} \frac{a_i^2}{\sum_{j \neq 0} a_j^2} \left(1 - \frac{\lambda_i}{d}\right)^s \tag{25}$$

where λ_i is the eigenvalue associated to the elementary function ϕ_i. In particular, for an elementary landscape $r(s) = (1 - \lambda/d)^s$, and the autocorrelation coefficient is $\xi = d/\lambda$. In general, if the landscape is not elementary we have the following result

$$r(1) = \frac{\sum_{i \neq 0} a_i^2 \left(1 - \frac{\lambda_i}{d}\right)}{\sum_{j \neq 0} a_j^2} = \frac{\sum_{i \neq 0} a_i^2 - \sum_{i \neq 0} a_i^2 \frac{\lambda_i}{d}}{\sum_{j \neq 0} a_j^2}$$
$$= 1 - \frac{\sum_{i \neq 0} a_i^2 \frac{\lambda_i}{d}}{\sum_{j \neq 0} a_j^2}$$

and the autocorrelation coefficient can be computed as

$$\xi = \frac{d \sum_{j \neq 0} a_j^2}{\sum_{i \neq 0} a_i^2 \lambda_i}. \tag{26}$$

The sum of the squared Fourier coefficients a_j^2 associated to the same eigenvalue λ_i is $|X|(\overline{f_i^2} - \overline{f_i}^2)$, where f_i is the sum of all the elementary components $a_i \phi_i$ with the same eigenvalue λ_i and the overline represents the average over the entire search space X.

In particular, for the general FAP we get the following expression for the autocorrelation coefficient

$$\xi = \left(B_r \frac{r}{d} + B_{2r} \frac{2r}{d}\right)^{-1} \tag{27}$$

where the values B_r and B_{2r} are defined as

$$B_r = \frac{r^n(\overline{f_r^2} - \overline{f_r}^2)}{r^n(\overline{f^2} - \overline{f}^2)} = \frac{\overline{f_r^2} - \overline{f_r}^2}{\overline{f^2} - \overline{f}^2} \tag{28}$$

$$B_{2r} = \frac{r^n(\overline{f_{2r}^2} - \overline{f_{2r}}^2)}{r^n(\overline{f^2} - \overline{f}^2)} = \frac{\overline{f_{2r}^2} - \overline{f_{2r}}^2}{\overline{f^2} - \overline{f}^2} \tag{29}$$

and are called the *amplitude* of the eigenvalues r and $2r$, respectively. B_r and B_{2r} are defined only in the case in which the objective function f is not a constant function (the denominator of B_λ is the variance of f in the search space). On the other hand, the expression $B_r + B_{2r} = 1$ holds (see [7] for more details).

The previous expressions allows to compute the exact autocorrelation coefficient of FAP using the landscape decomposition of the objective function. This is a theoretical application of the landscape decomposition shown in Section 4. In order to complete the computation we need a closed formula for $\sum_{x\in X} f_r^2$ and $\sum_{x\in X} f^2$. In vectorial form, the previous expressions are the squared norm of vectors f_r and f. Let us start with f. According to (10) we can write:

$$\begin{aligned}\sum_{x\in X} f^2 &= \sum_{x\in X} \sum_{i,j,i',j'=1}^{n} \sum_{p,q,p',q'=1}^{r} w_{i,j}^{p,q} w_{i',j'}^{p',q'} \varphi_{i,j}^{p,q}(x) \varphi_{i',j'}^{p',q'}(x) \\ &= \sum_{i,j,i',j'=1}^{n} \sum_{p,q,p',q'=1}^{r} w_{i,j}^{p,q} w_{i',j'}^{p',q'} \left(\sum_{x\in X} \delta_{x(i)}^{p} \delta_{x(j)}^{q} \delta_{x(i')}^{p'} \delta_{x(j')}^{q'} \right) \\ &= \sum_{i,j,i',j'=1}^{n} \sum_{p,q,p',q'=1}^{r} w_{i,j}^{p,q} w_{i',j'}^{p',q'} t(\{(i,p),(j,q),(i',p'),(j',q')\}) \end{aligned} \tag{30}$$

where we commuted the order of the sums, used the fact that $\varphi_{i,j}^{p,q}(x) = \delta_{x(i)}^{p} \delta_{x(j)}^{q}$, and introduced a new function t that is defined as

$$t : \mathcal{P}(V \times F) \to \mathbb{N} \tag{31}$$

$$S \mapsto t(S) = \sum_{x\in X} \prod_{(i,p)\in S} \delta_{x(i)}^{p}. \tag{32}$$

In (30) the function t is applied to $S = \{(i,p),(j,q),(i',p'),(j',q')\}$. However, it should be noticed that the number of elements of S is not always four, since some of their members (pairs) could coincide for some combinations of the mute variables i, j, i', etc. Although we have now an expression for $\sum_{x\in X} f^2$, this expression is not practical, since it includes a factor, t, that requires a summation over all the elements of X. In the following we are going to simplify the expression of t to make the computation feasible. We can observe that t is, in fact, a counting function. It is counting the number of elements in X that fulfill a given condition. Let us rewrite the definition of t as:

$$t(S) = \sum_{x\in X} \prod_{(i,p)\in S} \delta_{x(i)}^{p} = \sum_{x\in X} \mathit{True}\left(\bigwedge_{(i,p)\in S} x(i) = p \right) \tag{33}$$

where *True* is a function that maps a Boolean value to $\{0, 1\}$. It is 1 if the Boolean expression is true and 0 if it is false. The function t counts the number of solutions (elements in X) that fulfill the condition $\bigwedge_{(i,p)\in S} x(i) = p$. With some simple arguments we can find an alternative definition for t that is much easier to compute.

First, we must observe that if we find two tuples (i, p) and (j, q) in S such that $i = j$ and $p \neq q$, then the value of $t(S)$ must be zero because it is not possible to satisfy at the same time $x(i) = p$ and $x(j) = q$. We can characterize this situation using the condition $|\mathit{first}(S)| \neq |S|$, where $\mathit{first} : V \times F \to V$ is the function that maps a pair to its first element. That is, if the number of pairs in S is not equal to the number of first elements of these pairs, then there exist in S at least two pairs of the form (i, p) and (i, q) with $p \neq q$ and $t(S) = 0$.

Second, if the previous situation does not hold, that is $|\mathit{first}(S)| = |S|$, then the pairs in S set the value for $|S|$ components of the solution vector. The number of solutions in X with the fixed components is $t(S) = r^{n-|S|}$. Now, we are able to redefine t as:

$$t(S) = \begin{cases} r^{n-|S|} & \text{if } |\mathit{first}(S)| = |S| \\ 0 & \text{if } |\mathit{first}(S)| \neq |S|. \end{cases} \tag{34}$$

The new definition does not require any summation over X and it is computationally efficient ($O(|S|)$). This also makes the computation of $\sum_{x\in X} f^2$ efficient with a corresponding complexity of at most $O(n^4 r^4)$.

Let us continue now with the computation of $\sum_{x\in X} f_r^2$. Using (15) we can write:

$$\begin{aligned}\sum_{x\in X} f_r^2 = \sum_{x\in X} \Bigg(&\frac{1}{r^2} \sum_{\substack{i,j,i',j'=1 \\ i\neq j\, i'\neq j'}}^{n} \sum_{p,q,p',q'=1}^{r} w_{i,j}^{p,q} w_{i',j'}^{p',q'} \phi_{i,j,-2}^{p,q}(x) \phi_{i',j',-2}^{p',q'}(x) - \frac{2}{r} \sum_{\substack{i,j,i'=1 \\ i\neq j}}^{n} \sum_{p,q,p'=1}^{r} w_{i,j}^{p,q} w_{i',i'}^{p',p'} \phi_{i,j,-2}^{p,q}(x) \varphi_{i',i'}^{p',p'}(x) \\ &+ \sum_{i,i'=1}^{n} \sum_{p,p'=1}^{r} w_{i,i}^{p,p} w_{i',i'}^{p',p'} \varphi_{i,i}^{p,p}(x) \varphi_{i',i'}^{p',p'}(x) \Bigg)\end{aligned}$$

$$= \frac{1}{r^2} \sum_{\substack{i,j,i',j'=1 \\ i\neq j\, i'\neq j'}}^{n} \sum_{p,q,p',q'=1}^{r} w_{i,j}^{p,q} w_{i',j'}^{p',q'} \left(\sum_{x\in X} (\delta_{x(i)}^{p} + \delta_{x(j)}^{q})(\delta_{x(i')}^{p'} + \delta_{x(j')}^{q'}) \right)$$

$$+ \frac{2}{r} \sum_{\substack{i,j,i'=1 \\ i\neq j}}^{n} \sum_{p,q,p'=1}^{r} w_{i,j}^{p,q} w_{i',i'}^{p',p'} \left(\sum_{x\in X} (\delta_{x(i)}^{p} + \delta_{x(j)}^{q}) \delta_{x(i')}^{p'} \right) + \sum_{i,i'=1}^{n} \sum_{p,p'=1}^{r} w_{i,i}^{p,p} w_{i',i'}^{p',p'} \left(\sum_{x\in X} \delta_{x(i)}^{p} \delta_{x(i')}^{p'} \right).$$

If we use again the function t we can write:

$$\sum_{x\in X} f_r^2 = \frac{1}{r^2} \sum_{\substack{i,j,i',j'=1 \\ i\neq j\, i'\neq j'}}^{n} \sum_{p,q,p',q'=1}^{r} w_{i,j}^{p,q} w_{i',j'}^{p',q'} \left(\sum_{\substack{l\in \\ \{(i,p),(j,q)\}}} \sum_{\substack{l'\in \\ \{(i',p'),(j',q')\}}} t(l\cup l') \right)$$

$$+ \frac{2}{r} \sum_{\substack{i,j,i'=1 \\ i\neq j}}^{n} \sum_{p,q,p'=1}^{r} w_{i,j}^{p,q} w_{i',i'}^{p',p'} \left(\sum_{l\in\{(i,p),(j,q)\}} t(l\cup\{(i',p')\}) \right) + \sum_{i,i'=1}^{n} \sum_{p,p'=1}^{r} w_{i,i}^{p,p} w_{i',i'}^{p',p'} t(\{(i,p),(i',p')\}). \tag{35}$$

The previous expression suggests an algorithm for computing $\sum_{x\in X} f_r^2$ with complexity at most $O(n^4r^4)$. The expressions for $\overline{f}$ and $\overline{f_r}$ are shorter. The expression for $\overline{f}$ can be written as:

$$\overline{f} = \frac{1}{r^n} \sum_{x\in X} \sum_{i,j=1}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} \varphi_{i,j}^{p,q}(x) = \frac{1}{r^n} \sum_{i,j=1}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} \left(\sum_{x\in X} \delta_{x(i)}^{p} \delta_{x(j)}^{q} \right)$$

$$= \frac{1}{r^n} \sum_{i,j=1}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} t(\{(i,p),(j,q)\}). \tag{36}$$

The expression for $\overline{f_r}$ is:

$$\overline{f_r} = \frac{1}{r^n} \sum_{x\in X} \left(-\frac{1}{r} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} \phi_{i,j,-2}^{p,q}(x) + \sum_{i=1}^{n} \sum_{p=1}^{r} w_{i,i}^{p,p} \varphi_{i,i}^{p,p}(x) \right)$$

$$= \frac{1}{r^n} \left(\frac{1}{r} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} \sum_{x\in X} (\delta_{x(i)}^{p} + \delta_{x(j)}^{q}) + \sum_{i=1}^{n} \sum_{p=1}^{r} w_{i,i}^{p,p} \sum_{x\in X} \delta_{x(i)}^{p} \right)$$

$$= \frac{1}{r^n} \left(\frac{1}{r} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} (t(\{(i,p)\}) + t(\{(j,q)\})) + \sum_{i=1}^{n} \sum_{p=1}^{r} w_{i,i}^{p,p} t(\{(i,p)\}) \right)$$

$$= \frac{1}{r^n} \left(\frac{2r^{n-1}}{r} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} + r^{n-1} \sum_{i=1}^{n} \sum_{p=1}^{r} w_{i,i}^{p,p} \right)$$

$$= \frac{2}{r^2} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \sum_{p,q=1}^{r} w_{i,j}^{p,q} + \frac{1}{r} \sum_{i=1}^{n} \sum_{p=1}^{r} w_{i,i}^{p,p}. \tag{37}$$

Using the previous results we can state the following

Theorem 6. *The autocorrelation coefficient ξ of the landscape of any instance of the generalized FAP with the neighborhood operator defined in Section 4 can be computed in $O(n^4r^4)$.*

Proof. Eqs. (30), (35), (36) and (37) allows one to compute $\overline{f^2}, \overline{f_r^2}, \overline{f}$ and $\overline{f_r}$ using algorithms of complexity $O(n^4r^4)$, $O(n^4r^4)$, $O(n^2r^2)$ and $O(n^2r^2)$, respectively. The results can be combined in (28) to compute B_r and, hence, the value of $B_{2r} = 1 - B_r$. Finally, the autocorrelation coefficient ξ can be determined with (27). The final complexity of this computation of ξ from the data of the instance is, thus, $O(n^4r^4)$. □

Now let us compute the greatest lower bound and the least upper bound of ξ. We can write (27) in the following way:

$$\xi = \frac{n(r-1)}{r(2-B_r)} \tag{38}$$

where we used the fact that $B_{2r} + B_r = 1$. The coefficients B_λ are between 0 and 1. The minimum value for ξ is obtained when $B_r = 0$, where we have $\xi_{lb} = n(r-1)/(2r)$. The maximum value is reached when $B_r = 1$ and we have $\xi_{ub} = n(r-1)/r$. Then, for the FAP we have

$$\frac{n(r-1)}{2r} \leq \xi \leq \frac{n(r-1)}{r}. \tag{39}$$

The previous expression provides an upper and a lower bound for ξ. Furthermore, we know that these bounds can be reached in practice. According to Theorem 5, the lower bound ξ_{lb} is reached when condition (20) holds but condition (21) does not hold. That is, f is an elementary landscape with $\lambda = 2r$. Similarly, the upper bound ξ_{ub} is reached when condition (21) holds but condition (20) does not hold (f is an elementary landscape with $\lambda = r$). Thus, ξ_{lb} and ξ_{ub} are the greatest lower bound and the least upper bound of ξ, respectively. One interesting implication of this is that, fixing the values of n and r, the autocorrelation coefficient of any instance of graph coloring is lower than or equal to the autocorrelation coefficient of any instance of the generalized FAP.

7. Conclusions and future work

In this paper we have studied the Frequency Assignment Problem from the point of view of the landscapes' theory. We have proven that the general FAP can be written as a sum of two elementary landscapes. A practical application of this decomposition is the computation of the average value of the objective function in the neighborhood of a solution using only the value of the elementary components in that solution. We have given conditions under which the problem is an elementary landscape and we have highlighted some subclasses of the problem that are elementary. In particular, the graph coloring problem is an elementary subclass of FAP. The landscape decomposition of FAP has allowed us to provide a method for computing the autocorrelation coefficient for any instance of the problem in polynomial time. From a practical point of view, the autocorrelation coefficient is a tool that can be used to understand why one local search method is better than another for a given problem. Furthermore, it can be used to guide the design of new operators and search methods.

As future work we plan to develop a systematic methodology for decomposing objective functions into elementary landscapes using elementary concepts of linear algebra. Such a methodology could be useful for finding the decomposition of general landscapes. The previous work on this topic [7] is based on the Fourier expansion of the objective function in terms of a basis of eigenvectors of the Laplacian. Our plans are to build a methodology which does not require the Fourier expansion but it is based on the analysis of small instances of the problem and the generalization of this analysis. Furthermore, we expect the methodology to be partially mechanical, thus allowing the development of software tools for applying it. This methodology has partially been applied to obtain the results shown in this paper but their details have been omitted. We also plan to study the practical implications of all the theoretical results presented here. The improvement of search methods and the explanation of the behavior of some search algorithms are just two examples of possible applications of the theoretical results. In this paper we have provided some characterizations for the elementary landscapes and the sums of elementary landscapes that can be useful for decomposing other objective functions. We can study new problems and analyze their landscape decomposition in a similar way as we did for the generalized FAP in this paper.

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