Flujo de Ricci-Hamilton

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Serie de conferencias

1. **Demostración de la conjetura**
   - Lunes 16 de abril (10:30)
2. **Flujo de Ricci-Hamilton**
   - Viernes 20 de abril (10:00)
3. **Solitones de Ricci y singularidades**
   - Lunes 23 de abril (10:30)
4. **Aportaciones de Perelman**
   - Viernes 27 de abril (10:00)

Illustration by Robert Neubecker.

Demostración de la conjetura

- Geometría y curvatura riemanniana
- Flujo de Ricci como ecua. derivadas parcial.
- Otras ecuaciones de curvatura
- Principio del máximo, existencia y unicidad
- Pinzamiento de Hamilton-Ivey

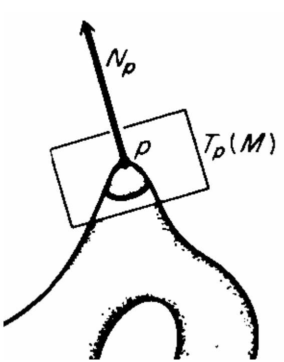
Geometría riemanniana

- Espacio euclídeo: longitudes y ángulos medidos mediante un producto escalar

$$\mathbf{x}\cdot\mathbf{y}=\sum_{i=1}^{n}x_iy_i=x_1y_1+x_2y_2+\cdots+x_ny_n. \qquad \|\mathbf{x}\|=\sqrt{\mathbf{x}\cdot\mathbf{x}}=\sqrt{\sum_{i=1}^{n}(x_i)^2}. \qquad \theta=\cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right)$$

- Un producto escalar general : B es matriz simétrica

$$\mathbf{u}^T\mathbf{B}\mathbf{u}=\sum_{i,j=1}^{n}B_{ij}u^iu^j$$



Definition 1.1. Let M be an n-dimensional manifold. A *Riemannian metric* g on M is a smooth section of $T^*M\otimes T^*M$ defining a positive definite symmetric bilinear form on T_pM for each $p\in M$. In local coordinates $(x^1,\cdots,x^n)$, one has a natural local basis $\{\partial_1,\cdots,\partial_n\}$ for TM, where $\partial_i=\frac{\partial}{\partial x^i}$. The metric tensor $g=g_{ij}dx^i\otimes dx^j$ is represented by a smooth matrix-valued function

$$g_{ij}=g(\partial_i,\partial_j).$$

The pair (M,g) is a *Riemannian manifold*. We denote by (g^{ij}) the inverse of the matrix (g_{ij}).

Geometría riemanniana

Riemannian geometry / Manfredo do Carmo

2. Riemannian Metrics

2.1 DEFINITION. A *Riemannian metric* (or *Riemannian structure*) on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $\langle\ ,\ \rangle_p$ (that is, a symmetric, bilinear, positive-definite form) on the tangent space T_pM, which varies differentiably in the following sense: If $\mathbf{x}: U \subset \mathbf{R}^n \to M$ is a system of coordinates around p, with $\mathbf{x}(x_1, x_2, \ldots, x_n) = q \in \mathbf{x}(U)$ and $\frac{\partial}{\partial x_i}(q) = d\mathbf{x}_q(0, \ldots, 1, \ldots, 0)$, then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q)\rangle_q = g_{ij}(x_1, \ldots, x_n)$ is a differentiable function on U.

curve c to a closed interval $[a, b] \subset I$

$$\ell_a^b(c) = \int_a^b \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2} dt.$$

volume $\mathrm{vol}(R)$ of R

$$\mathrm{vol}(R) = \int_{\mathbf{x}^{-1}(R)} \sqrt{\det(g_{ij})}\, dx_1 \ldots dx_n$$

Geometría riemanniana

2. Affine Connections

2.1 DEFINITION. An *affine connection* ∇ on a differentiable manifold M is a mapping

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$$

which is denoted by $(X,Y) \xrightarrow{\nabla} \nabla_X Y$ and which satisfies the following properties :

i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$.

iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$,

in which $X, Y, Z \in \mathcal{X}(M)$ and $f, g \in \mathcal{D}(M)$.

$$\nabla : \text{Vector Fields} \otimes_{\mathbb{R}} \text{Vector Fields} \rightarrow \text{Vector Fields}$$

$$\nabla(X \otimes Y) = \nabla_X(Y)$$

$$X = \sum_i x_i X_i, \qquad X_i = \frac{\partial}{\partial x_i},$$

$$\boxed{\nabla_{X_i} X_j = \textstyle\sum_k \Gamma_{ij}^k X_k,}$$

vector field V along the differentiable curve $c: I \rightarrow M$

vector field $\frac{DV}{dt}$ along c, called the covariant derivative of

$V(t) = Y(c(t))$, then $\frac{DV}{dt} = \nabla_{dc/dt} Y$.

Geometría riemanniana

3.2 PROPOSITION. *Let M be a Riemannian manifold. A connection ∇ on M is compatible with a metric if and only if for any vector fields V and W along the differentiable curve $c: I \to M$ we have*

$$\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle, \quad t \in I. \tag{3}$$

3.6 Theorem. *(Levi-Civita). Given a Riemannian manifold M, there exists a unique affine connection ∇ on M satisfying the conditions:*

a) ∇ is symmetric.
b) ∇ is compatible with the Riemannian metric.

the functions Γ^k_{ij} defined on U by $\nabla_{X_i} X_j = \sum_k \Gamma^k_{ij} X_k$, the *coefficients of the connection* ∇ on U or the *Christoffel symbols* of the connection. From (9) it follows that

$$\Gamma^m_{ij} = \frac{1}{2} \sum_k \left\{ \frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right\} g^{km}.$$

Curvatura riemanniana (Gauss)

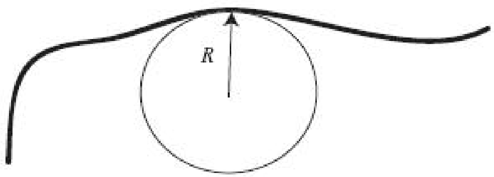


Figure 1: 2D Curvature.

$$\kappa = 1/R$$

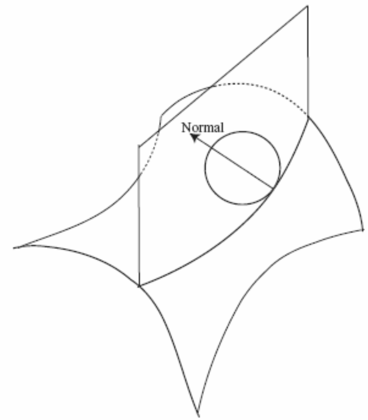


Figure 2: Curvature in 3D

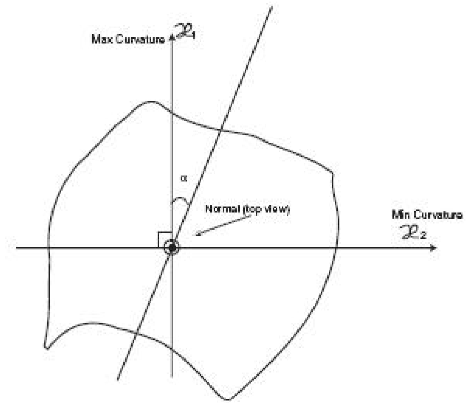


Figure 3: Curvature in 3D, top view

A *normal curve* is the intersection of a surface with a plane containing the normal n. For a given direction d in the tangent plane there is a unique normal curve, obtained by intersecting the plane spanned by n and d with the plane.

The curvature of a normal curve is called *sectional curvature.*

The *principal curvatures* (κ_1, κ_2) are the maximal and minimal sectional curvatures. The *principal curvature directions* are the directions in the tangent plane for which the maximum and minimum are attained. These directions are perpendicular to each other.

$$\kappa = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha$$

Gaussian curvature

$$\kappa_G = \kappa_1 \kappa_2$$

Curvatura riemanniana

2. Curvature

2.1 DEFINITION. The *curvature* R of a Riemannian manifold M is a correspondence that associates to every pair $X, Y \in \mathcal{X}(M)$ a mapping $R(X,Y): \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad Z \in \mathcal{X}(M),$$

where ∇ is the Riemannian connection of M.

Observe that if $M = R^n$, then $R(X,Y)Z = 0$ for all X, Y, Z

$$\boxed{R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = (\nabla_{\partial/\partial x_j} \nabla_{\partial/\partial x_i} - \nabla_{\partial/\partial x_i} \nabla_{\partial/\partial x_j})\frac{\partial}{\partial x_k}}$$

$$R(X_i, X_j)X_k = \sum_\ell R^\ell_{ijk} X_\ell$$

As frequently happens in mathematics, a "workable" formulation of the concept of curvature required a long time for its development. When such a formulation finally appeared it had the advantage of being easy to use to prove theorems but it had the disadvantage of being so far removed from the initial intuitive concept

Curvatura riemanniana

Definition 1.5. The *sectional curvature* of a 2-plane $P \subset T_pM$ is defined as

$$K(P) = \mathcal{R}(X, Y, X, Y),$$

where $\{X, Y\}$ is an orthonormal basis of P. We say that (M, g) has *positive sectional curvature* (resp., *negative sectional curvature*) if $K(P) > 0$ (resp., $K(P) < 0$) for every 2-plane P. There are analogous notions of non-negative and non-positive sectional curvature.

In local coordinates, suppose that $X = X^i\partial_i$ and $Y = Y^i\partial_i$. Then we have

$$K(P) = R_{ijkl}X^iY^jX^kY^l.$$

A riemannian manifold is said to have *constant sectional curvature* if $K(P)$ is the same for all $p \in M$ and all two-planes $P \subset T_pM$. One can show that a manifold (M, g) has constant sectional curvature λ if and only if

$$R_{ijkl} = \lambda(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Of course, the sphere of radius r in $\mathbb{R}^n$ has constant sectional curvature $1/r^2$

Tensores de Curvatura

Levi-Civita connection $$\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

Riemannian curvature tensor $$R_{ijl}^{k} = \frac{\partial \Gamma_{jl}^{k}}{\partial x^i} - \frac{\partial \Gamma_{il}^{k}}{\partial x^j} + \Gamma_{ip}^{k} \Gamma_{jl}^{p} - \Gamma_{jp}^{k} \Gamma_{il}^{p}$$

$$R_{ijkl} = g_{kp} R_{ijl}^{p} \qquad R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}$$

Bianchi identity $$R_{ijkl} + R_{jkil} + R_{kijl} = 0$$

Ricci tensor is the contraction $$R_{ik} = g^{jl} R_{ijkl}$$

scalar curvature $$R = g^{ij} R_{ij}.$$

Ricci flow of Hamilton $$\frac{\partial g_{ij}}{\partial t} = -2 R_{ij}$$

Curvatura de Riemann

- Tensor de curvatura de Ricci

$$Ric(X,Y) = Ric_g(X,Y) = g^{kl} R(X, \partial_k, Y, \partial_l).$$

$$Ric = Ric_{ij} dx^i \otimes dx^j, \quad Ric_{ij} = Ric(\partial_i, \partial_j).$$

- Escalar de curvatura

$$R = R_g = \mathrm{tr}_g Ric = g^{ij} Ric_{ij}.$$

- R determina Ric y Riemann en n=2
- Ric determina Riemann en n=3
- Riemann es necesario sólo en n>3

Programa de Hamilton-Yau

- Ecuación del flujo de Ricci (el volumen decrece)

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g)$$

$$R_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{lower order terms}$$

(operador Laplace-Beltrami)

- Ecuación del flujo de Ricci normalizado (volumen const.)

$$\frac{\partial g}{\partial t} = -\mathrm{Ric} + \frac{R}{2} g$$

Ecuación débilmente parabólica

$$\frac{\partial u}{\partial t} = a_{ij}\partial_i\partial_j u + b_i\partial_i u + cu$$ *parabolic* if a_{ij} is uniformly positive definite

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

Consider the Laplace Beltrami operator

$$\begin{aligned}\Delta &= \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right)\\ &= g^{ij}\partial_i\partial_j + \text{lower order terms}\end{aligned}$$

$$g^{ij}\xi_i\xi_j = |\xi|^2 > 0.$$

nonlinear PDE *parabolic* if the *linearisation* is parabolic

$$\frac{\partial v}{\partial t} = P(v)$$

$$\frac{\partial v}{\partial t} = [DP(w)]\, v$$

Análisis dimensional

$$\Gamma = O(g^{-1}\partial g)$$

$$\mathrm{Riem}, \mathrm{Ric} = O(g^{-1}\partial^2 g) + O(g^{-2}(\partial g)(\partial g))$$

$$R = O(g^{-2}\partial^2 g) + O(g^{-3}(\partial g)(\partial g)).$$

Laplace-Beltrami operator $\Delta_g f = O(g^{-1}\partial^2 f) + O(g^{-2}(\partial g)(\partial f)).$

Ricci flow equation $\partial_t g = O(g^{-1}\partial^2 g) + O(g^{-2}(\partial g)(\partial g))$

where the lead term $O(g^{-1}\partial^2 g)$ is not elliptic.

This is not a manifestly parabolic equation.

PERELMAN'S PROOF OF THE POINCARÉ CONJECTURE: A NONLINEAR PDE PERSPECTIVE

TERENCE TAO

Solución Flujo de Ricci

Einstein metrics

$$R_{ij}(x,0) = \lambda g_{ij}(x,0), \quad \forall x \in M \qquad R_{ij}(x,t) = R_{ij}(x,0) = \lambda g_{ij}(x,0)$$

$$\boxed{\lambda > 0} \qquad g_{ij}(x,t) = \rho^2(t) g_{ij}(x,0)$$

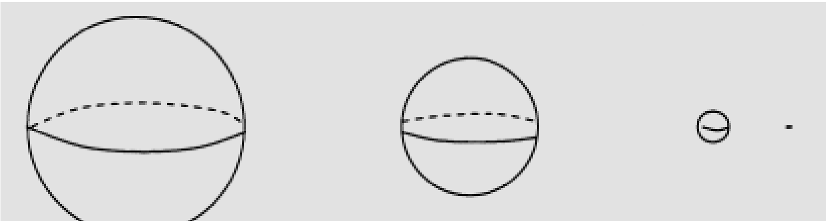
$$\frac{\partial(\rho^2(t) g_{ij}(x,0))}{\partial t} = -2\lambda g_{ij}(x,0)$$

$$\frac{d\rho}{dt} = -\frac{\lambda}{\rho}, \qquad \rho^2(t) = 1 - 2\lambda t.$$

metric $g_{ij}(x,t)$ shrinks homothetically to a $\boxed{\text{point as } t \to T = 1/2\lambda}$

as $t \to T$, the scalar curvature becomes infinite like $1/(T-t)$.

- Solución "explota" en tiempo finito (blow-up)



Solución Flujo de Ricci

Einstein metrics

$$\boxed{R_{ij}(x,0) = -\lambda g_{ij}(x,0)} \quad \forall x \in M$$

$$\lambda < 0 \qquad g_{ij}(x,t) = \rho^2(t) g_{ij}(x,0)$$

$$\frac{d\rho}{dt} = \frac{\lambda}{\rho}, \qquad \rho^2(t) = 1 + 2\lambda t.$$

metric $g_{ij}(x,t)$ exists and expands homothetically for all times,
and the curvature will fall back to zero like $-1/t$

evolving metric $g_{ij}(x,t)$ only goes back in time to $-1/2\lambda$,
the metric explodes out of a single point in a "big bang".

- Solución ha "explotado" en el "pasado"

Existencia-Unicidad de Solución

THEOREM 1.2.2 (Hamilton [58], De Turck [43]). *Let* $(M, g_{ij}(x))$ *be a compact Riemannian manifold. Then there exists a constant* $T > 0$ *such that the initial value problem*

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \\ g_{ij}(x,0) = g_{ij}(x) \end{cases}$$

has a unique smooth solution $g_{ij}(x,t)$ *on* $M \times [0,T)$.

Hamilton [16], uses the Nash-Moser iteration method to compensate for the lack of smoothing present in the equation. The second approach, due to de Turck [14], is a "gauge fixing" approach, taking advantage of the geometric nature of the Ricci flow, which in practical terms creates the "gauge invariance" formed by diffeomorphisms of the manifold (i.e. changes of coordinates).

$$\begin{cases} \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) + \nabla_i V_j + \nabla_j V_i, \\ g_{ij}(x,0) = \overset{o}{g}_{ij}(x), \end{cases}$$

$$V_i = g_{ik} g^{jl} (\Gamma^k_{jl} - \overset{o}{\Gamma}{}^k_{jl}),$$

$$\partial_t g = \Delta_g g + O(g^{-2}(\partial g)(\partial g)).$$

obviously a parabolic equation

Evolución de las Curvaturas

Proposition 1.3.1 (Hamilton [58]). *Under the Ricci flow (1.1.5), the curvature tensor satisfies the evolution equation*

$$\frac{\partial}{\partial t}R_{ijkl} = \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql})$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ *and* Δ *is the Laplacian with respect to the evolving metric.*

Proof. Choose $\{x^1, \ldots, x^m\}$ to be a normal coordinate system at a fixed point. At this point, we compute

$$\frac{\partial}{\partial t}\Gamma^h_{jl} = \frac{1}{2}g^{hm}\left\{\frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial t}g_{lm}\right) + \frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial t}g_{jm}\right) - \frac{\partial}{\partial x^m}\left(\frac{\partial}{\partial t}g_{jl}\right)\right\}$$

$$= \frac{1}{2}g^{hm}(\nabla_j(-2R_{lm}) + \nabla_l(-2R_{jm}) - \nabla_m(-2R_{jl})),$$

$$\frac{\partial}{\partial t}R^h_{ijl} = \frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial t}\Gamma^h_{jl}\right) - \frac{\partial}{\partial x^j}\left(\frac{\partial}{\partial t}\Gamma^h_{il}\right),$$

$$\frac{\partial}{\partial t}R_{ijkl} = g_{hk}\frac{\partial}{\partial t}R^h_{ijl} + \frac{\partial g_{hk}}{\partial t}R^h_{ijl}.$$

$$\frac{\partial}{\partial t}Rm = \Delta Rm + Rm * Rm.$$

Evolución de las Curvaturas

COROLLARY 1.3.2. *The Ricci curvature satisfies the evolution equation*

$$\frac{\partial}{\partial t} R_{ik} = \Delta R_{ik} + 2g^{pr} g^{qs} R_{piqk} R_{rs} - 2g^{pq} R_{pi} R_{qk}.$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t} R_{ik} &= g^{jl} \frac{\partial}{\partial t} R_{ijkl} + \left(\frac{\partial}{\partial t} g^{jl} \right) R_{ijkl} \\
&= g^{jl} [\Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\
&\quad - g^{pq} (R_{pjkl} R_{qi} + R_{ipkl} R_{qj} + R_{ijpl} R_{qk} + R_{ijkp} R_{ql})] \\
&\quad - g^{jp} \left(\frac{\partial}{\partial t} g_{pq} \right) g^{ql} R_{ijkl} \\
&= \Delta R_{ik} + \boxed{2g^{jl} (B_{ijkl} - 2B_{ijlk})} + 2g^{pr} g^{qs} R_{piqk} R_{rs} \\
&\quad - 2g^{pq} R_{pk} R_{qi}.
\end{aligned}$$

Evolución de las Curvaturas

COROLLARY 1.3.3. *The scalar curvature satisfies the evolution equation*

$$\boxed{\frac{\partial R}{\partial t} = \Delta R + 2|\mathrm{Ric}\,|^2.}$$

Proof.

$$\begin{aligned}\frac{\partial R}{\partial t} &= g^{ik}\frac{\partial R_{ik}}{\partial t} + \left(-g^{ip}\frac{\partial g_{pq}}{\partial t}g^{qk}\right)R_{ik} \\ &= g^{ik}(\Delta R_{ik} + 2g^{pr}g^{qs}R_{piqk}R_{rs} - 2g^{pq}R_{pi}R_{qk}) + 2R_{pq}R_{ik}g^{ip}g^{qk} \\ &= \Delta R + 2|\mathrm{Ric}\,|^2.\end{aligned}$$

- Principio "débil" del máximo (ec. parabólica)

PROPOSITION 2.1.2. *If the scalar curvature R of the solution $g_{ij}(t), 0 \le t \le T$, to the Ricci flow is nonnegative at $t = 0$, then it remains so on $0 \le t \le T$.*

$$\frac{\partial R}{\partial t} = \Delta R + 2\,|Rc|^2 \ge \Delta R.$$

Evolución de las Curvaturas

- Principio del máximo

if $R_{min}(0) > 0$, $$R_{min}(t) \geq \frac{1}{R_{min}(0)^{-1} - \frac{2}{3}t}$$

Theorem 4. *Let M be a 3-dimensional compact manifold with a Riemannian metric of strictly positive Ricci curvature. Then M has a Riemannian metric of constant strictly positive sectional curvature.*

Remark 5. *By a classical result, M is then isometric to a finite quotient of the round S^3. One says that M is a spherical manifold. If M is simply connected, M is diffeomorphic to S^3. This is the first step on the way to the Poincaré conjecture.*

Pinzamiento de Hamilton-Ivey

2.4. Hamilton-Ivey Curvature Pinching Estimate. The Hamilton-Ivey curvature pinching estimate roughly says that if a solution to the Ricci flow on a three-manifold becomes singular (i.e., the curvature goes to infinity) as time t approaches the maximal time T, then the most negative sectional curvature will be small compared to the most positive sectional curvature.

Hamilton-Ivey pinching estimate THEOREM 2.4.1 (Hamilton [63], Ivey [73]). *Assume at $t = 0$ the eigenvalues $\lambda \geq \mu \geq \nu \geq -1$.*

$$R \geq (-\nu)[\log(-\nu) - 3], \qquad \textit{whenever } \nu < 0.$$

- Autovalores del tensor de Ricci

eigenvalues $\lambda \geq \mu \geq \nu$

$$\begin{pmatrix} \lambda & & \\ & \mu & \\ & & \nu \end{pmatrix}$$

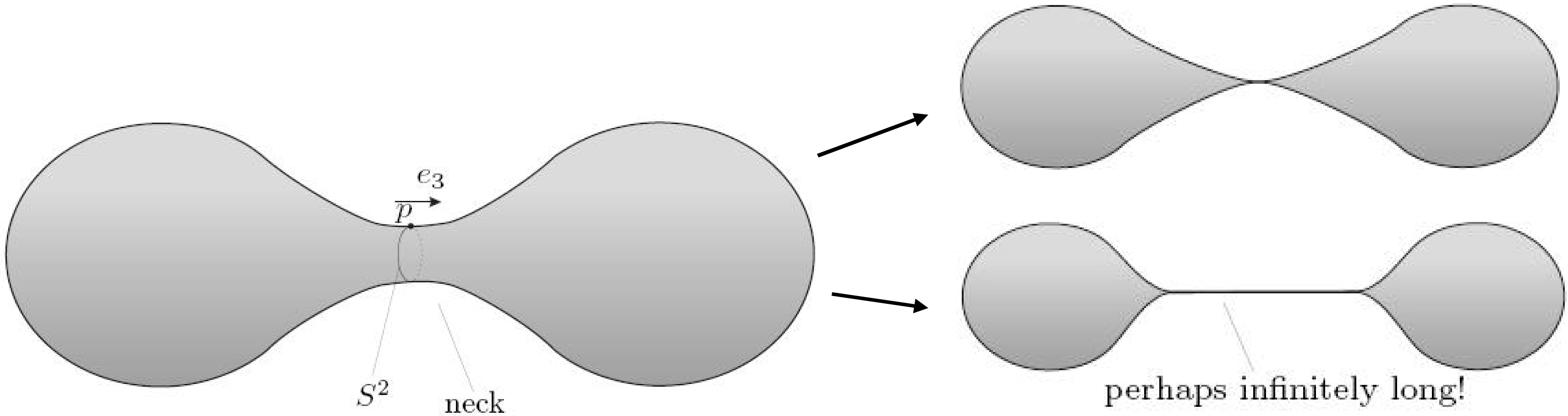
The scalar curvature $R = \lambda + \mu + \nu$

ODE corresponding to PDE

$$\begin{cases} \frac{d}{dt}\lambda = \lambda^2 + \mu\nu, \\ \frac{d}{dt}\mu = \mu^2 + \lambda\nu, \\ \frac{d}{dt}\nu = \nu^2 + \lambda\mu. \end{cases}$$

$$\frac{d}{dt}(\lambda + \mu + \nu) \geq 0$$

Pinzamiento de Hamilton-Ivey



$$\mathrm{Ric}(e_1, e_1) = K_{e_1 \wedge e_2} + K_{e_1 \wedge e_3} = \text{very positive}$$
$$\mathrm{Ric}(e_2, e_2) = K_{e_2 \wedge e_1} + K_{e_2 \wedge e_3} = \text{very positive}$$
$$\mathrm{Ric}(e_3, e_3) = K_{e_3 \wedge e_1} + K_{e_3 \wedge e_2} = \text{slightly negative}$$

To control $|Rm|$ in dimension 3, it suffices to bound R from above. I.e., if $R \leq C$, then $|Rm| \leq C$. (Since $R \leq C$ implies $|\nu| \leq C$ which implies $|Rm| \leq C$.)

Zoom de las soluciones

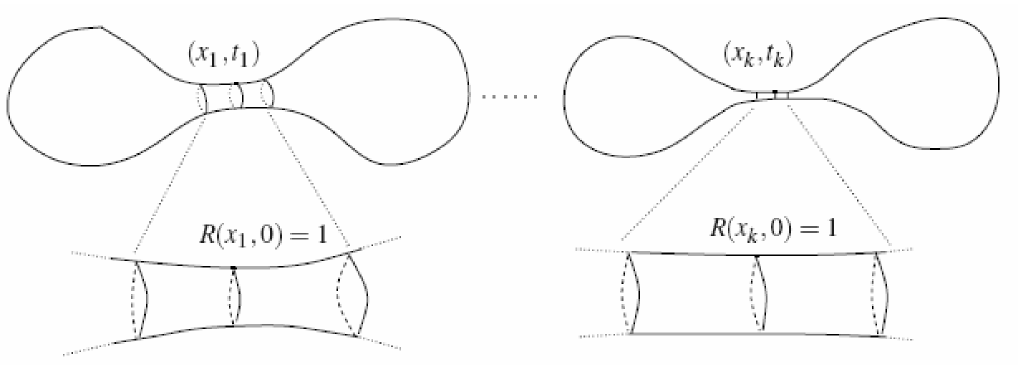
- Reescalado parabólico del flujo de Ricci

$$h(t) = \lambda^2 g(\lambda^{-2} t)$$

More precisely, let $g(t)$ be a Ricci flow on $M \times [0,T)$, $x_0 \in M$, $t_0 \in [0,T)$ such that $R(x,t) \leq Q_0 \equiv R(x_0,t_0)$ for all $x \in M$ and $t \leq t_0$. Then

$$g_0(t) = Q_0 g\left(t_0 + \frac{t}{Q_0}\right)$$

is a Ricci flow on $[-t_0Q_0, (T-t_0)Q_0)$ and $R_0(x,t) \leq 1$ for any $x \in M$ and $t \leq 0$. This is called a parabolic rescaling of $g(t)$



sequence of parabolic rescalings