Solitones del Flujo de Ricci y Formación de Singularidades

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LENGUAJES Y CIENCIAS DE LA COMPUTACIÓN UNIVERSIDAD DE MÁLAGA

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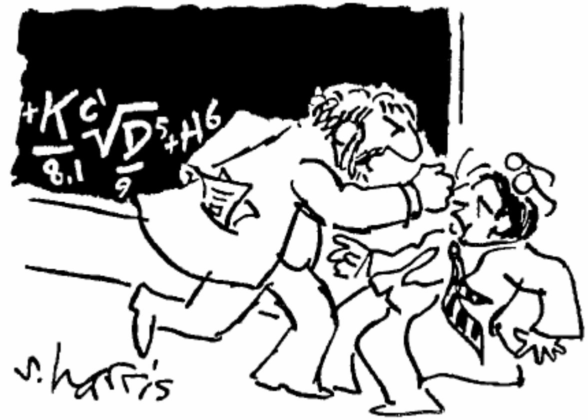
Serie de conferencias

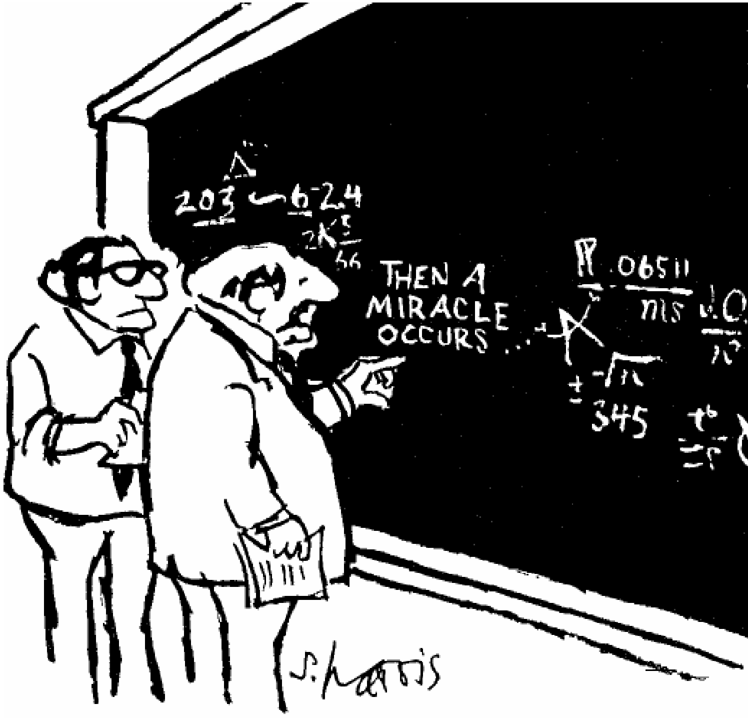
1. **Demostración de la con**
    - Lunes 16 de abril (10:30)
2. **Flujo de Ricci-Hamilton**
    - Viernes 20 de abril (10:00)
3. **Solitones de Ricci y singularidades**
    - Lunes 23 de abril (10:30)
4. **Aportaciones de Perelman**
    - Viernes 27 de abril (10:00)



"YOU WANT PROOF? I'LL GIVE YOU PROOF

Demostración de la conjetura

- Solitones en general
- Blow-up en general
- Solitones de Ricci
- Tipos y modelos de singularidades
- Soluciones ancianas



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

Ondas solitarias o "solitones"

- Soluciones auto-semejantes

$$u = B\tau^{\mu} U(\xi / A\tau^{\lambda}) \qquad u = u_0(\tau) U(\xi / \xi_0(\tau)) .$$

- Ondas solitarias (travelling wave solutions)

$$v = V(x - X(t)) + V_0(t) \qquad v = V(x - \lambda t + c) .$$

$$v = \ln u, \quad t = \ln \tau, \quad V_0(t) = \ln u_0(\tau) ,$$
$$V = \ln U, \quad x = \ln \xi, \quad X(t) = \ln \xi_0(\tau) ,$$

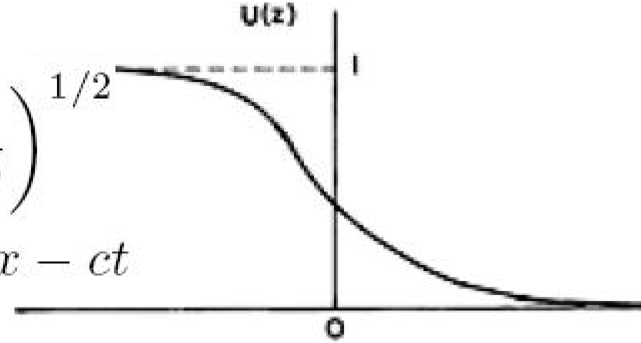
- Un ejemplo "sencillo"

$$\phi_t - a\phi_{xx} + b\phi^n = 0, \qquad n = 2 \text{ and } a = b = 1.$$

$$\phi(x, t) = \frac{G(\tau)}{R_1 - t} \qquad \tau = \frac{R_2 - \frac{1}{2}x}{\sqrt{R_1 - t}} \qquad G_{\tau\tau} - 2\tau G_\tau - 4(G + G^2) =$$

Ondas solitarias o "solitones"

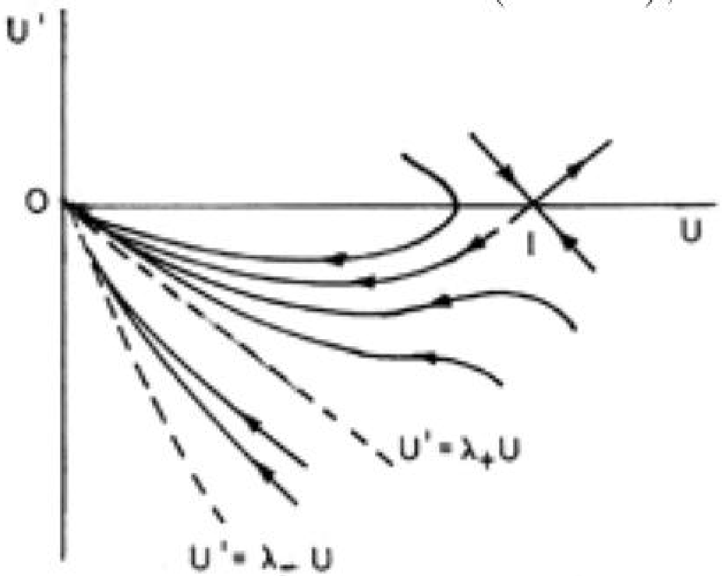
- Ejemplo: Ecuación de Fisher



$$u_t = ku(1-u) + Du_{xx} \qquad t^* = kt, \quad x^* = x\left(\frac{k}{D}\right)^{1/2}$$

$$u_t = u(1-u) + u_{xx} \qquad u(x,t) = U(z), \quad z = x - ct$$

$$-cU' - U'' = U(1-U), \quad ' = \frac{d}{dz}. \qquad \begin{aligned} U' &= V \\ V' &= -cV - U(1-U), \end{aligned}$$



Linearising the system the eigenvalues

$$P : (0,0) \qquad \lambda_\pm = \frac{1}{2}\left[-c \pm (c^2-4)^{1/2}\right],$$

stable node if $c^2 \geq 4$ stable spiral if $c^2 <$

$$Q : (1,0) \qquad \lambda_\pm = \frac{1}{2}\left[-c \pm (c^2+4)^{1/2}\right]$$

always a saddle point

Blow-up de soluciones

- Ecuaciones diferenciales ordinarias

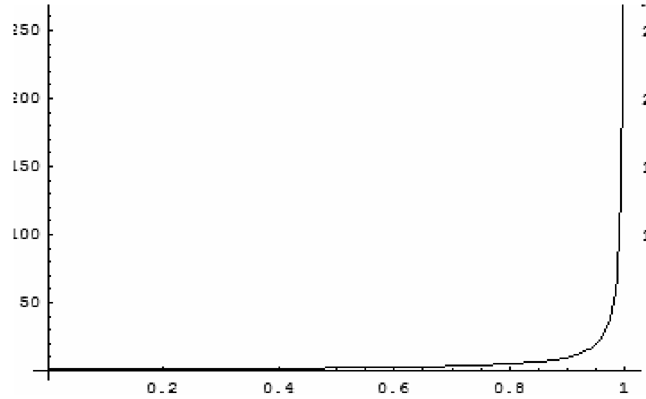
$$\boxed{Y_t = Y^2,}\ t > 0; \quad Y(0) = a. \tag{7.21}$$

Si el dato inicial es $a > 0$, se sigue immediatamente que existe una única solución definida en un intervalo temporal $0 < t < T$ con $T = 1/a$, y dada por la fórmula

$$\boxed{Y(t) = \frac{1}{T - t}.} \tag{7.22}$$

Vemos pues que la evolución está descrita por una función regular para $t < T$. Cuando $t \to T^-$ (límite por la izquierda), vemos que la solución explota, $Y(t) \to \infty$. No solo eso, también sabemos cual es la tasa de crecimiento cerca de la explosión, $Y(t) = O((T - t)^{-1})$. Este será para nosotros el *ejemplo elemental de explosión.*

- Solución con asíntota vertical



Blow-up de soluciones

- Explosión de soluciones : típico EDP NL parabólicas

We consider the problem of the blow-up of solutions of the initial value problem

$$u_t = u_{xx} + u^p \qquad (1)$$

where $p > 1$, $u = u(x,t)$, $x \in \mathbb{R}$, and $u(\cdot, 0) = u_0 \in C^0(\mathbb{R})$. It is well-known that, for a large class of initial data u_0, the solution will diverge in a finite time at a single point (for reviews on this problem, see [9, 17]).

We are interested in the profile of the solution at the time of blow-up. To explain what this means, let us fix the blow-up point to be 0 and the blow-up time to be T. Then, we ask whether it is possible to find a function $f^*(x)$ and a rescaling $g(t,T)$ so that

$$\lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} u(g(t,T)x, t) = f^*(x). \qquad (2)$$

$u(t)$ solves the ODE $\dot{u} = u^p$, i.e. $u(t) = ((p-1)(T-t))^{\frac{1}{1-p}}$ for $T = (p-1)^{-1} u_0^{1-p}$.

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} \varphi\left(\frac{x}{(T-t)^{1/2k}}, -\log(T-t)\right).$$

$$\dot{\varphi} = L_\tau^{-2} \varphi'' - \frac{1}{2k} \xi \varphi' - \frac{1}{p-1} \varphi + \varphi^p$$

$$\varphi(\xi, \tau_0) = T^{\frac{1}{p-1}} u_0(T^{\frac{1}{2k}} \xi)$$

Solitones de Ricci

- Soluciones auto-semejantes del Flujo de Ricci

$\lambda > 0,$ scaling time by λ and distances by $\lambda^{\frac{1}{2}}$,

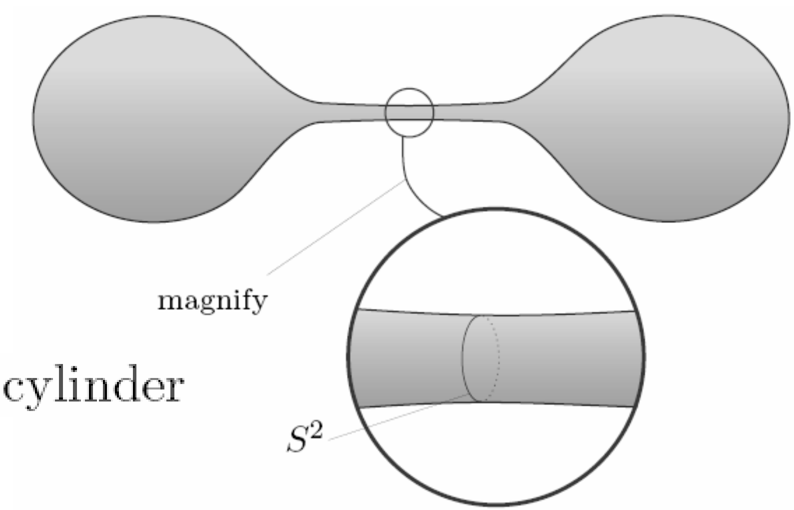
$$\boxed{\hat{g}(x,t) = \lambda g(x, t/\lambda),}$$

$$\frac{\partial \hat{g}}{\partial t}(x,t) = \frac{\partial g}{\partial t}(x, t/\lambda) = -2\mathrm{Ric}(g(t/\lambda))(x) = -2\mathrm{Ric}(\hat{g}(t))(x)$$

- Escalado: tensor de Ricci invariante
- Curvaturas se reescalan

$$\boxed{R(\hat{g}(x,t)) = \lambda^{-1} R(g(x, t/\lambda)).}$$

Límite puede ser no compacto
(topología original se pierde)



the blow-up looks like a part of the cylinder $S^2 \times \mathbb{R}$ (a 'neck')

Solución Flujo de Ricci

Einstein metrics

$$R_{ij}(x,0) = \lambda g_{ij}(x,0), \quad \forall x \in M \qquad R_{ij}(x,t) = R_{ij}(x,0) = \lambda g_{ij}(x,0)$$

$$\boxed{\lambda > 0} \qquad g_{ij}(x,t) = \rho^2(t) g_{ij}(x,0)$$

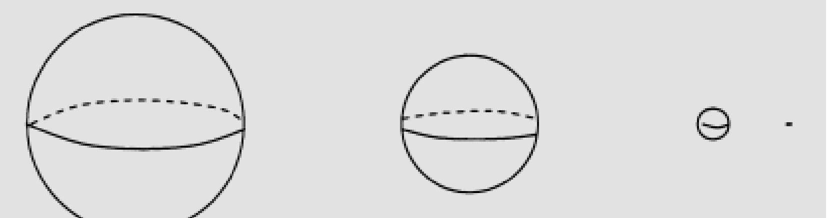
$$\frac{\partial(\rho^2(t) g_{ij}(x,0))}{\partial t} = -2\lambda g_{ij}(x,0)$$

$$\frac{d\rho}{dt} = -\frac{\lambda}{\rho}, \qquad \rho^2(t) = 1 - 2\lambda t.$$

metric $g_{ij}(x,t)$ shrinks homothetically to a $\boxed{\text{point as } t \to T = 1/2\lambda}$

as $t \to T$, the scalar curvature becomes infinite like $1/(T-t)$.

- Solución "explota" en tiempo finito (blow-up)



Solución Flujo de Ricci

Einstein metrics

$$\boxed{R_{ij}(x,0) = -\lambda g_{ij}(x,0)} \quad \forall x \in M$$

$$\lambda < 0 \qquad g_{ij}(x,t) = \rho^2(t) g_{ij}(x,0)$$

$$\frac{d\rho}{dt} = \frac{\lambda}{\rho}, \qquad \rho^2(t) = 1 + 2\lambda t.$$

metric $g_{ij}(x,t)$ exists and expands homothetically for all times,
and the curvature will fall back to zero like $-1/t$

evolving metric $g_{ij}(x,t)$ only goes back in time to $-1/2\lambda$,
the metric explodes out of a single point in a "big bang".

- Solución ha "explotado" en el "pasado"

Solitones de Ricci

- Solución auto-semejante módulo difeomorfimos

family of diffeomorphisms ψ_t $\boxed{\hat{g}(t) = \sigma(t)\psi_t^*(g(t)),}$

$\psi_t^*(g(t)) = \psi_t^*(g(t) - g(s)) + \psi_t^*(g(s))$ and differentiate at $t = s$

$$\frac{\partial \hat{g}}{\partial t} = \sigma'(t)\psi_t^*(g) + \sigma(t)\psi_t^*\left(\frac{\partial g}{\partial t}\right) + \sigma(t)\psi_t^*(\mathcal{L}_X g).$$

$$g(t) = g_0 \qquad \sigma(t) := 1 - 2\lambda t, \qquad X(t) := \tfrac{1}{\sigma(t)}Y,$$

$$\frac{\partial \hat{g}}{\partial t} = \sigma'(t)\psi_t^*(g_0) + \sigma(t)\psi_t^*(\mathcal{L}_X g_0) = \psi_t^*(-2\lambda g_0 + \mathcal{L}_Y g_0)$$

$$= \psi_t^*(-2\mathrm{Ric}(g_0)) = -2\mathrm{Ric}(\psi_t^* g_0) = -2\mathrm{Ric}(\hat{g}).$$

- Solitón de Ricci $\boxed{-2\mathrm{Ric}(g_0) = \mathcal{L}_Y g_0 - 2\lambda g_0}$

Definition 1.2.2. Such a flow is called a steady, expanding or shrinking 'Ricci soliton' depending on whether $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$ respectively.

$$-2R_{ij} = 2\lambda g_{ij} + \nabla_i V_j + \nabla_j V_i.$$

Solitones de Ricci

- Solitón gradiente de Ricci

$$-2\mathrm{Ric}(g_0) = \mathcal{L}_Y g_0 - 2\lambda g_0$$

$$-2R_{ij} = 2\lambda g_{ij} + \nabla_i V_j + \nabla_j V_i.$$

$$V_i = \nabla_i f$$

$$\boxed{R_{ij} + \lambda g_{ij} + \nabla_i \nabla_j f = 0.}$$

- Solitón "cigarro" (cigar) de Hamilton

$$g_{ij} = \frac{\delta_{ij}}{1 + x^2 + y^2} \qquad f(x,y) = (x^2 + y^2)/2, \qquad R_{ij} + \nabla_i \nabla_j f = 0$$

metric is rotationally symmetric, noncompact steady gradient Ricci soliton

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2} = \frac{dr^2 + r^2\ d\theta^2}{1 + r^2} = ds^2 + \tanh^2 s\, d\theta^2$$

- Tiende a cilindro de radio 1

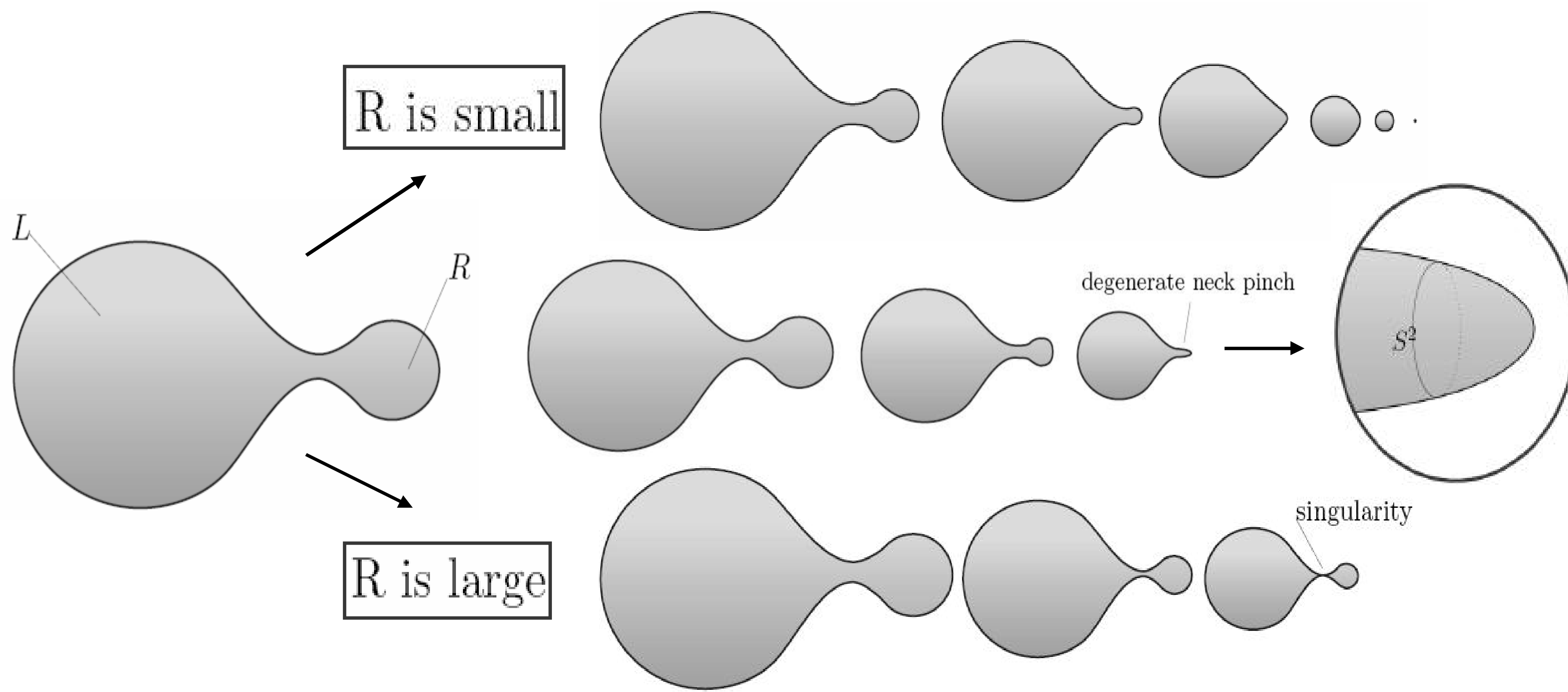
(R tiende a cero al crecer r)

$$R = \frac{4}{1 + r^2} = \frac{4}{\cosh^2 s} = \frac{16}{(e^s + e^{-s})^2}.$$

Witten Black Hole.

Solitones de Ricci

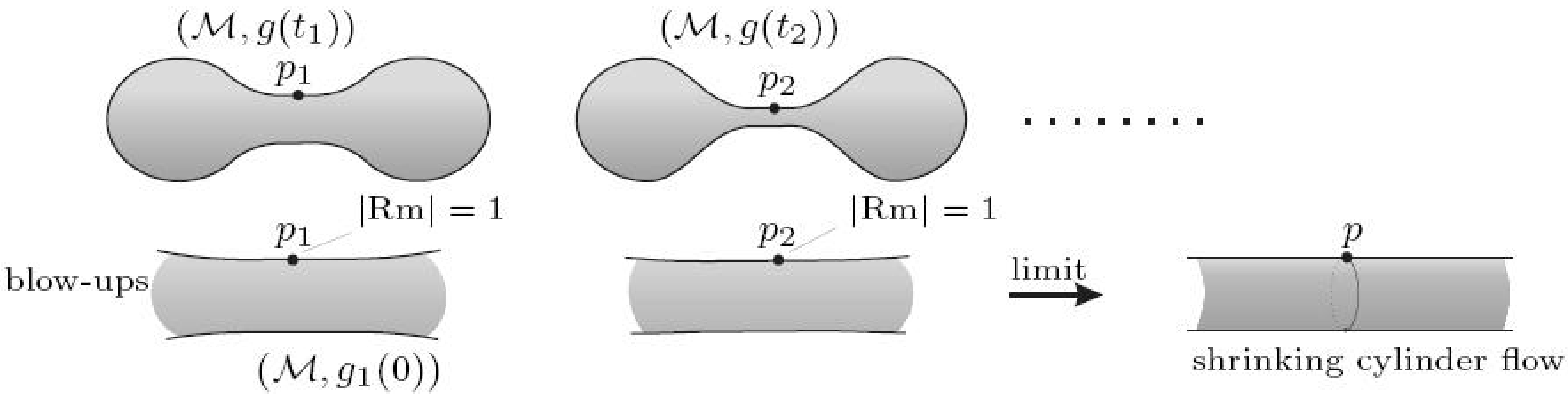
Pinzamiento del cuello (neck-pinch)



Solitón de Bryant (esferas de radio decrece como parábola)

Desarrollo de singularidades

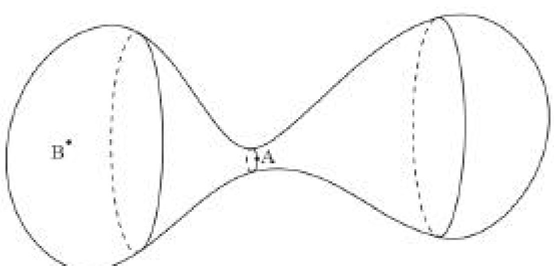
- Escalado dependiente del tiempo



- Convergencia de variedades riemannianas puntuadas

Definition 8.1. *A sequence $(\mathcal{M}_i, g_i, p_i)$ of smooth, complete, pointed Riemannian manifolds is said to* ***converge*** *smoothly to the smooth, complete, pointed manifold $(\mathcal{M}, g, p)$ as $i \to \infty$ if the*

- Requiere acotar el radio de inyectividad



The ***injectivity radius*** $\mathrm{inj}(p)$ *at a point $p \in \mathcal{M}$ is defined by*

$$\mathrm{inj}(p) := \sup\{r > 0 : \exp_p : B(0, r) \to \mathcal{M} \text{ is injective}\}.$$

Desarrollo de singularidades

Proposition 14.6 *Let* $(M^3, g(t))$ *be a complete ancient solution of the Ricci flow. Assume that there exists a continuous function* $\phi(t)$ *such that* $|K_{sec}(g_t)| \leq \phi(t)$. *Then* g_t *has nonnegative sectional curvature for as long as it exists.*

$$R_{\min}(t) \geq \frac{n}{n(R_{\min}(t_0))^{-1} - 2(t - t_0)} \geq -\frac{n}{2(t - t_0)} \text{ for all } t > t_0.$$

$$R_{min}(t) \geq -\lim_{t_0 \to -\infty} \frac{n}{2(t - t_0)} = 0.$$

Proposition 14.7 *In dimension* $n = 3$ *every Type I limit of Type I singularity has nonnegative sectional curvature for as long as it exists.*

Theorem 14.13 *(Perelman's No Local Collapsing Theorem). Let* $(M^n, g(t))$ *be a solution to the Ricci flow that becomes singular in a finite time* T. *Then there exists a constant* $C > 0$ *independent of* t *and a subsequence* (x_i, t_i) *such that*

$$inj(x_i, t_i) \geq \frac{C}{\sqrt{\max_M |Rm(\cdot, t)|}}.$$

Desarrollo de singularidades

Theorem 8.3. *Suppose that $(\mathcal{M}, g(t))$ is a solution of the Ricci flow defined on a maximal time interval $[0, T)$, where $T < \infty$. Then there exist points $p_i \in \mathcal{M}$ and times $t_i \in [0, T)$, $t_i \to T$ such that*

$$M_i := |\mathrm{Rm}|(p_i, t_i) = \sup_{x \in \mathcal{M}, t \in [0, t_i]} |\mathrm{Rm}|(x, t) \to \infty.$$

If we now define

$$\boxed{g_i(t) = M_i g\left(t_i + \frac{t}{M_i}\right)} \qquad \left(\text{for } t \le 0 \qquad |\mathrm{Rm}(g_i(t))| = \frac{|\mathrm{Rm}\left(g\left(t_i + \frac{t}{M_i}\right)\right)|}{M_i} \le \frac{M_i}{M_i} = 1\right)$$

then there exists $b > 0$ such that $(\mathcal{M}, g_i(t), p_i)$ converges to a Ricci flow $(\mathcal{N}, g_\infty(t), p_\infty)$ defined for $t \in (-\infty, b)$. Furthermore $\boxed{|\mathrm{Rm}(g_\infty(0))|(p_\infty) = 1}$ *and $|\mathrm{Rm}(g_\infty(t))| \le 1$ for $t \le 0$.*

the limit flow is defined for $t \in (-\infty, b)$ – such solutions are called **ancient**.

Theorem 6.8 *Let M be a compact Riemannian 3-manifold which admits a solution of the Ricci flow which develops a singularity at time T. Then there is a sequence of dilations[20] of the solution which converges to a quotient by isometries of one of the following manifolds:*

i) S^3, *(a topological space form)*

ii) $S^2 \times \mathbb{R}$, *(a neck)*

iii) $\Sigma \times \mathbb{R}$, *where Σ is a cigar solution,*

Hamilton 1995

Desarrollo de singularidades

Hamilton (1995)

Singularities		
Finite Time $T < \infty$	Type I	$\sup_{M\times[0,T)} \lvert Rm(\cdot,t)\rvert(T-t) < \infty$
	Type II$_a$	$\sup_{M\times[0,T)} \lvert Rm(\cdot,t)\rvert(T-t) = \infty$
Infinite Time $T = \infty$	Type II$_b$	$\sup_{M\times[0,\infty)} \lvert Rm(\cdot,t)\rvert t = \infty$
	Type III	$\sup_{M\times[0,\infty)} \lvert Rm(\cdot,t)\rvert t < \infty$

Singularity model	time interval	curvature bound
Ancient Type I	$(-\infty,\omega)/\omega > 0$	$\sup_{M_\infty\times(-\infty,0]} \lvert Rm_\infty(\cdot,t)\rvert \lvert t\rvert < \infty$
Ancient Type II	$(-\infty,\omega)/\omega > 0$	$\sup_{M_\infty\times(-\infty,0]} \lvert Rm_\infty(\cdot,t)\rvert \lvert t\rvert = \infty$
Eternal Type II	$(-\infty,\infty)$	$\sup_{M_\infty\times(-\infty,0]} \lvert Rm_\infty(\cdot,t)\rvert < \infty$
Immortal Type III	$(-\alpha,\infty)/\alpha > 0$	$\sup_{M_\infty\times(-\infty,0]} \lvert Rm_\infty(\cdot,t)\rvert \cdot t < \infty$

Soluciones ancianas

definition 1.1. *$(M, g(t))$ is a κ-solution if*

Perelman

- *$g(t)$ is an ancient solution of the Ricci flow*

$$\frac{\partial}{\partial t} g(t) = -2Ric_{g(t)}, \qquad -\infty < t \leq 0.$$

- *for each t, $g(t)$ is a complete, non flat metric of bounded curvature and non negative curvature operator.*

- *for each t, $g(t)$ is κ-noncollapsed on all scales, i.e. if $|Rm(g(t))| \leq \frac{1}{r^2}$ on $B = B(p,t,r)$, then*

$$\frac{vol_{g(t)}(B)}{r^n} \geq \kappa$$

Exemples: S^3 and $\mathbb{S}^2 \times \mathbb{R}$ with their standard flow are κ-solutions for some $\kappa > 0$. But $\mathbb{S}^2 \times \mathbb{S}^1$ with the standard flow is not a κ-solution. It is κ-collapsed at very negative times.

- All curvatures of $g(t)$ at x are controlled by the scalar curvature $R(x,t)$.
- For each point x in M, $R(x,t)$ is nondecreasing.

normalized κ-solution at (x_0, t_0) by a shift in time and a parabolic rescaling such that $R_{g_0}(x_0, 0) = 1$.

Soluciones ancianas

Asymptotic solitons Perelman defines an asymptotic soliton $(M_{-\infty}, g_{-\infty}, x_{-\infty})$ of an n-dimensional κ-solution $(M, g(t)$ as follows. Pick a sequence $t_k \to -\infty$.

theorem 1.3 ([P03]I.11.2). *there exists $x_k \in M$ such that $(M, \frac{1}{-t_k} g(t_k - t_k t), x_k)$ (sub) converge to a non flat gradient shrinking soliton$(M_{-\infty}, g_{-\infty}, x_{-\infty})$, called an asymptotic soliton of the κ-solution.*

corollary 1.4 (of the compactness theorem). *Any 3-dimensional asymptotic soliton is a κ-solution.*

classification of κ-solutions We have the following

theorem 1.9. *Any κ-solution $(M, g(t))$ is diffeomorphic to one of the following.*

a $\mathbb{S}^2 \times \mathbb{R}$ *or* $\mathbb{S}^2 \times_{\mathbb{Z}_2} \mathbb{R} = \mathbb{RP}^3 - \overline{\mathbb{B}^3}$, *and $g(t)$ is the round cylindrical flow.*

b $\mathbb{R}^3$ *and $g(t)$ has strictly positive curvature.*

c *A finite isometric quotient of the round S^3 and $g(t)$ has positive curvature. Moreover, $g(t)$ is round if and only if the asymptotic soliton is compact. If the asymptotic soliton is non compact, M is diffeomorphic to $\mathbb{S}^3$ or $\mathbb{RP}^3$.*

Soluciones ancianas

Perelman

Recall that an **ε-neck** (of radius r) is an open set with a Riemannian metric, which is, after scaling the metric with factor r^{-2}, ε-close to the standard neck $\mathbb{S}^2 \times \mathbb{I}$ with the product metric, where $\mathbb{S}^2$ has constant scalar curvature one and $\mathbb{I}$ has length $2\varepsilon^{-1}$ and the ε-closeness refers to the $C^{[\varepsilon^{-1}]}$ topology.

A metric on $\mathbb{S}^2 \times \mathbb{I}$, such that each point is contained in some ε-neck, is called an **ε-tube**, or an **ε-horn**, or a **double ε-horn**, if the scalar curvature stays bounded on both ends, or stays bounded on one end and tends to infinity on the other, or tends to infinity on both ends, respectively.

A metric on $\mathbb{B}^3$ or $\mathbb{RP}^3 \setminus \bar{\mathbb{B}}^3$ is called a **ε-cap** if the region outside some suitable compact subset is an ε-neck. A metric on $\mathbb{B}^3$ or $\mathbb{RP}^3 \setminus \bar{\mathbb{B}}^3$ is called an **capped ε-horn** if each point outside some compact subset is contained in an ε-neck and the scalar curvature tends to infinity on the end.

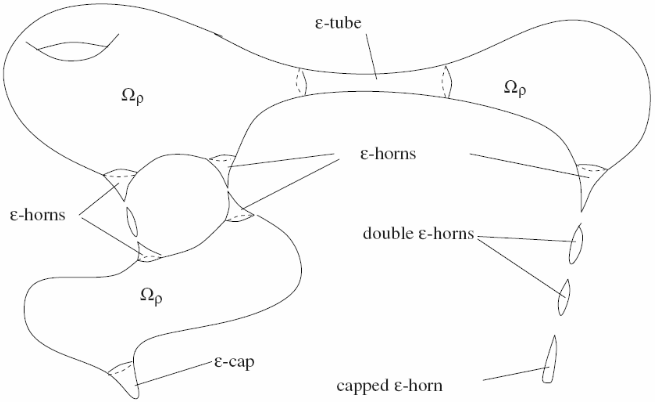
(a) an ε-tube with boundary components in Ω_ρ, or

(b) an ε-cap with boundary in Ω_ρ, or

(c) an ε-horn with boundary in Ω_ρ, or

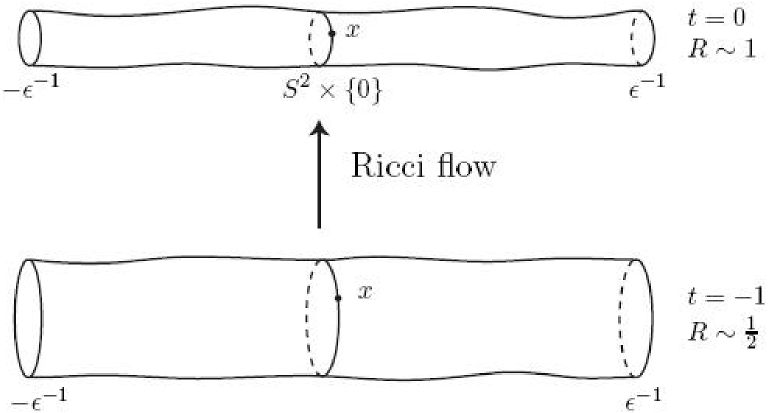
(d) a capped ε-horn, or

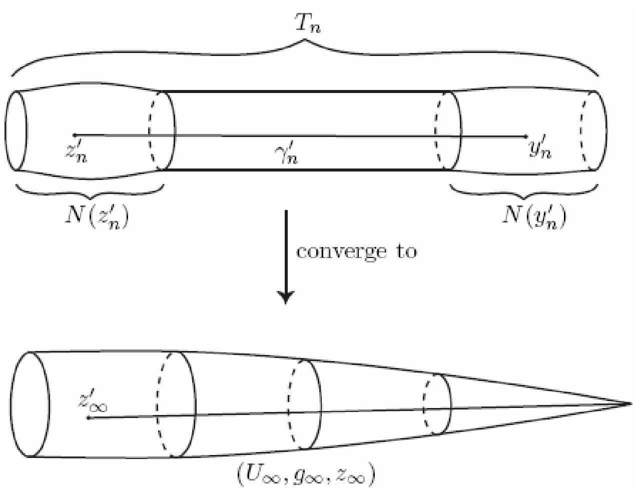
(e) a double ε-horn.

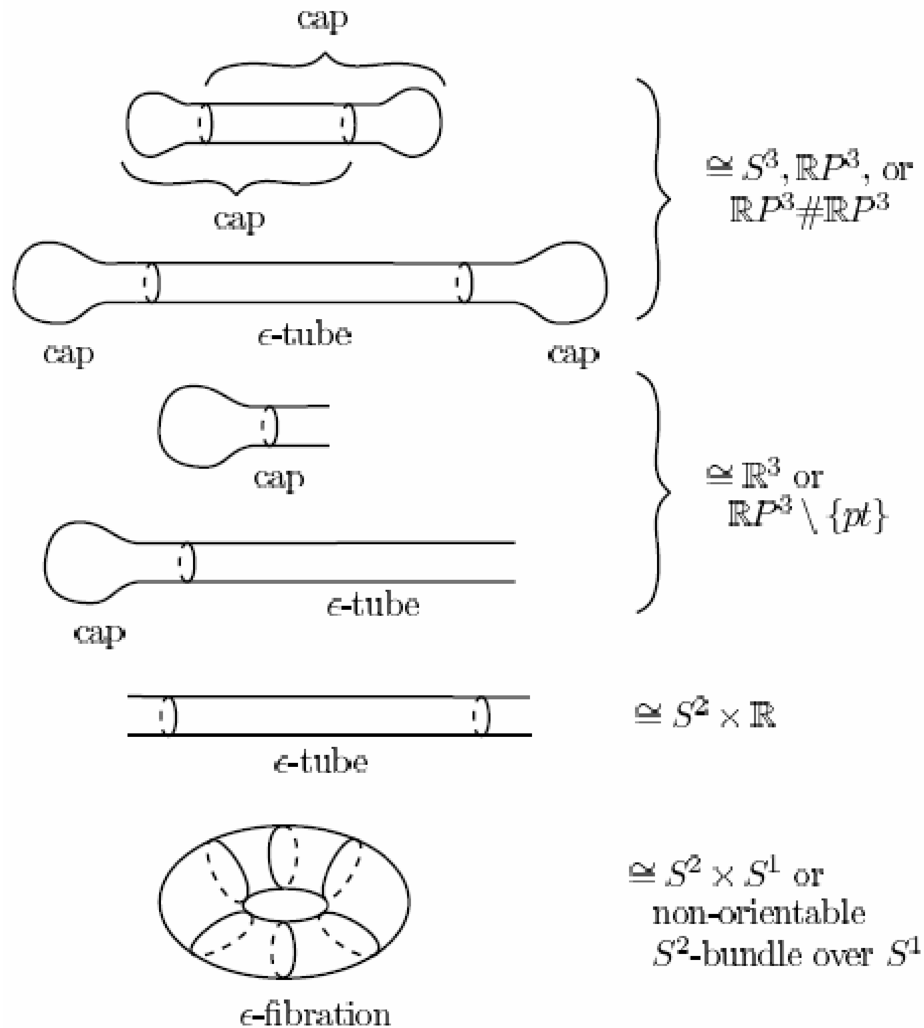


It is clear that there is a definite lower bound (depending on ρ) for the volume of subsets of types (a), (b) and (c), so there can be only a finite number of them. Thus we conclude that there is only a finite number of components of Ω containing points of Ω_ρ, and every such component has a finite number of ends, each being an ε-horn.

Soluciones ancianas







Flujo de Ricci con cirugía (idea)